ECE 313: Problem Set 2

Due: Tuesday September 6 at 11:59 p.m.
Reading: ECE 313 Course Notes, Sections 2.1–2.4.2

1. **[Distribution of number of matches]**
   Suppose five people write their names on slips of paper; the slips of paper are randomly shuffled and then each person gets back one slip of paper; all possibilities of who gets what slip are equally likely. Let \( X \) denote the number of people who get back the slip with their own name.

   (a) Describe the sample space \( \Omega \) for the experiment. How many elements does it have?
   (b) Find the probability mass function of \( X \).
   (c) Find \( E[X] \) and \( E[X^2] \).
   (d) Find \( \text{Var}(X) \).
   (e) Find \( \text{Var}(2X + 1) \).

2. **[Mean, Variance, LOTUS]**
   Consider a random variable \( X \).

   (a) Is it possible for the mean of \( X \) to be 5 and the standard deviation 0? If so, construct a pmf for \( X \) that corresponds to these values. If not, prove the impossibility.
   (b) Is it possible for the mean of \( X \) to be 0, the second moment 1, and the standard deviation 1? If so, construct a pmf for \( X \) that corresponds to these values. If not, prove the impossibility.
   (c) Suppose the mean of \( X \) is \( \mu \) and the standard deviation is \( \sigma \). Find a formula for the mean of the random variable \( Y = 3X^2 + 6X - 1 \) in terms of \( \mu \) and \( \sigma \).

3. **[First and second moments of a ternary random variable]**
   This problem focuses on the possible mean and variance of a random variable \( X \) with support set \( \{-1, 0, 1\} \). Let \( p_X(-1) = a \) and \( p_X(1) = b \) and \( p_X(0) = 1 - a - b \). Let \( \mu = E[X] \), \( \sigma^2 = \text{Var}(X) \), and \( m_2 = E[X^2] \).

   (a) What are the conditions for \( a \) and \( b \) to make the pmf valid?
   (b) Find \((a, b)\) so that \( \mu = \frac{1}{2} \) and \( \sigma = \frac{1}{2} \).
   (c) Express \( a \) and \( b \) in terms of \( \mu \) and \( m_2 \). Determine and sketch the region of \((\mu, m_2)\) pairs for which there is a valid choice of \((a, b)\).
   (d) Determine and sketch the set of \((\mu, \sigma^2)\) pairs for which there is a valid choice of \((a, b)\).

4. **[Conditional probability]**
   Two fair dice are rolled.

   (a) Find the conditional probability that the difference is 2 given the product is odd.
   (b) Find the conditional probability that at least one die lands on 2, given that the product is even.