University of Illinois **Fall 2022**

ECE 313: Problem Set 12: Problems and Solutions

Due: *Monday* December 3 at 11:59 pm
Reading: *ECE 313 Course Notes*, Sections

1. **[Estimation]**

   (a) **Solution:** To solve for c, we use $\int\int f_{X,Y}(x,y)\ dxdy = 1$.

   $$1 = \int_0^{1/2}\int_0^{1/2} c(x+y)\ dxdy = \frac{c}{8} \implies c = 8.$$

   (b) **Solution:** We can calculate the marginal distribution f_X and the conditional distribution $f_{Y|X}$ as follows.

   $$f_X(x) = \int_0^{1/2} 8(x+y)\ dy = 1 + 4x$$
   $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{8(x+y)}{1+4x}, \quad x, y \in \left[0, \frac{1}{2}\right]$$

   By the definition of the unconstrained estimator g_*,

   $$g_*(u_0) = E[Y|X = u_0] = \int y f_{Y|X}(y|u_0)\ dy = \int \frac{y \cdot 8(u_0+y)}{1+4u_0}\ dy = \frac{3u_0+1}{3+12u_0}$$

   (c) **Solution:** From the lecture, we know that the linear estimator is given by:

   $$L^*(u_0) = E[Y] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(u_0 - E[X])$$

   Since X, Y are symmetric in $f_{X,Y}$, $E[X] = E[Y]$ and $\mathrm{Var}(X) = \mathrm{Var}(Y)$.

   $$E[X] = \int_0^{1/2} x(1+4x)\ dx = \frac{7}{24}$$
   $$\mathrm{Var}(X) = E\left[X^2\right] - (E[X])^2 = \frac{11}{576}$$
   $$E[XY] = \int\int xy f_{X,Y}(x,y)\ dxdy = \frac{1}{12}$$
   $$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = -\frac{1}{576}$$

   Plugging in the values yields $L^*(u_0) = \frac{1}{11}\left(\frac{7}{2} - u_0\right)$.

2. **[Gaussian random variables]**

   (a) **Solution:** Using the bi-linearity of covariance,

   $$\mathrm{Cov}(X-Y, 2X+3Y-1) = 2\,\mathrm{Var}(X) + \mathrm{Cov}(X,Y) - 3\,\mathrm{Var}(Y)$$

   Since X, Y are independent, $\mathrm{Cov}(X,Y) = 0$ and hence $\mathrm{Cov}(X-Y, 2X+3Y-1) = -1$.

   (b) **Solution:** We use the fact that linear combinations of independent Gaussian random variables are also Gaussian. It is easy to see that $E[2X+Y] = 0$, and

   $$\mathrm{Var}(2X+Y) = \mathrm{Cov}(2X+Y, 2X+Y) = 4\,\mathrm{Var}(X) + 4\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y) = 5$$

   This shows that $2X+Y \sim N(0,5)$ and

   $$P(2X+Y \geq 1) = P\left(\frac{2X+Y}{\sqrt{5}} \geq \frac{1}{\sqrt{5}}\right) = Q\left(\frac{1}{\sqrt{5}}\right).$$

(c) **Solution:** Similarly, $X - Y \sim N(0, 2)$ since $E[X - Y] = 0$ and

$$\text{Var}(X - Y) = \text{Var}(X) - 2\,\text{Cov}(X, Y) + \text{Var}(Y) = 2$$

Then since $P(X - Y > a) = P(X - Y < -a)$ for $a > 0$ by symmetry, we have

$$P\left((X - Y)^2 > 4\right) = 2P(X - Y > 2) = 2P\left(\frac{X - Y}{\sqrt{2}} > \sqrt{2}\right) = 2Q\left(\sqrt{2}\right)$$

3. **[Gaussian Random variables and Estimation]**

(a) **Solution:** Using the linearity of expectation and bi-linearity of covariance, we get

$$E[W] = E[2X + Y - 1] = 2E[X] + E[Y] - 1 = 0$$
$$\text{Var}(W) = \text{Cov}(2X + Y - 1, 2X + Y - 1) = 4\,\text{Var}(X) + 4\,\text{Cov}(X, Y) + \text{Var}(Y) = 37$$

Here we use $\text{Cov}(X, Y) = \rho\,\sigma_X \sigma_Y = 3$.

(b) **Solution:** First, observe that W, Y are also jointly Gaussian since W is a linear combination of jointly Gaussian X, Y. From the lecture, we know that the best mean square error estimator for jointly Gaussian random variables is the linear MSE, i.e.

$$g_*(w_0) = E[Y] + \frac{\text{Cov}(W, Y)}{\text{Var}(W)}(w_0 - E[W])$$

Since $\text{Cov}(W, Y) = \text{Cov}(2X + Y - 1, Y) = 2\,\text{Cov}(X, Y) + \text{Var}(Y) = 15$, $g_*(w_0) = -1 + \frac{15}{37}w_0$, and the resulting MSE is

$$E\left[(Y - g_*(W))^2\right] = \text{Var}[Y]\left(1 - \rho^2_{W,Y}\right) = \frac{9 \cdot 12}{37}$$

4. **[LLN]**

Solution: Let $Y = \frac{S_n}{n} - \mu$. It is easy to check that $E[Y] = 0$ and $\text{Var}(Y) = \frac{\sigma^2}{n}$. Since Y is a linear combination of i.i.d. Gaussian R.V.'s, Y is also Gaussian, i.e. $Y \sim N\left(0, \frac{\sigma^2}{n}\right)$. So we have

$$P(|Y| \geq \epsilon) = P\left(\frac{|Y|}{\sigma/\sqrt{n}} \geq \frac{\epsilon}{\sigma/\sqrt{n}}\right) = 2Q\left(\frac{\epsilon\sqrt{n}}{\sigma}\right)$$

Let $X \sim N(0, 1)$, and for an interval I denote $\mathbb{1}_I$ be the function defined as

$$\mathbb{1}_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{otherwise} \end{cases}$$

To prove the limit,

$$\begin{aligned}
\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) &= \lim_{n\to\infty} Q\left(\frac{\epsilon\sqrt{n}}{\sigma}\right) \\
&= \lim_{n\to\infty} \int_{\frac{\epsilon\sqrt{n}}{\sigma}}^{\infty} f_X(x)\ dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} \mathbb{1}_{\left[\frac{\epsilon\sqrt{n}}{\sigma}, \infty\right)}(x)\, f_X(x)\ dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} \mathbb{1}_{(-\infty, x]}\left(\frac{\epsilon\sqrt{n}}{\sigma}\right) f_X(x)\ dx \\
&= \int_{-\infty}^{\infty} \lim_{n\to\infty} \mathbb{1}_{(-\infty, x]}\left(\frac{\epsilon\sqrt{n}}{\sigma}\right) f_X(x)\ dx
\end{aligned}$$

Since $\lim_{n\to\infty} \mathbb{1}_{(-\infty,x]}\left(\frac{\epsilon\sqrt{n}}{\sigma}\right) = 0$ for a fixed x, it follows

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) = \int_{-\infty}^{\infty} 0 \cdot f_X(x)\, dx = 0$$

Alternatively, we can use Chebyshev's inequality to get the upper bound

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2/n}{\epsilon^2}$$

Since ϵ, σ are fixed, taking the limit on the both sides proves $P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0$ as $n \to \infty$.