ECE 313: Problem Set 12: Problems and Solutions

Due: Monday December 3 at 11:59 pm
Reading: ECE 313 Course Notes, Sections

1. [Estimation]
   
   (a) **Solution:** To solve for $c$, we use
   \[ 1 = \int_{0}^{1/2} \int_{0}^{1/2} c(x + y) \, dx \, dy = \frac{c}{8} \implies c = 8. \]

   (b) **Solution:** We can calculate the marginal distribution $f_X$ and the conditional distribution $f_{Y|X}$ as follows.
   \[ f_X(x) = \int_{0}^{1/2} 8 (x + y) \, dy = 1 + 4x \]
   \[ f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{8(x+y)}{1+4x}, \quad x, y \in [0, \frac{1}{2}] \]

   By the definition of the unconstrained estimator $g_*$,
   \[ g_*(u_0) = E[Y|X = u_0] = \int y f_{Y|X}(y|u_0) \, dy = \int \frac{y \cdot 8(u_0 + y)}{1+4u_0} \, dy = \frac{3u_0 + 1}{3+12u_0} \]

   (c) **Solution:** From the lecture, we know that the linear estimator is given by:
   \[ L^*(u_0) = E[Y] + \frac{\text{Cov}(X,Y)}{\text{Var}(X)} (u_0 - E[X]) \]

   Since $X, Y$ are symmetric in $f_{X,Y}$, $E[X] = E[Y]$ and $\text{Var}(X) = \text{Var}(Y)$.
   \[ E[X] = \int_{0}^{1/2} x (1 + 4x) \, dx = \frac{7}{24} \]
   \[ \text{Var}(X) = E[X^2] - (E[X])^2 = \frac{11}{576} \]
   \[ E[XY] = \int \int xy f_{X,Y}(x,y) \, dx \, dy = \frac{1}{12} \]
   \[ \text{Cov}(X,Y) = E[XY] - E[X]E[Y] = -\frac{1}{576} \]

   Plugging in the values yields $L^*(u_0) = \frac{7}{12} (u_0) - u_0$.

2. [Gaussian random variables]
   
   (a) **Solution:** Using the bi-linearity of covariance,
   \[ \text{Cov}(X - Y, 2X + 3Y - 1) = 2 \text{Var}(X) + \text{Cov}(X,Y) - 3 \text{Var}(Y) \]

   Since $X, Y$ are independent, $\text{Cov}(X,Y) = 0$ and hence $\text{Cov}(X - Y, 2X + 3Y - 1) = -1$.

   (b) **Solution:** We use the fact that linear combinations of independent Gaussian random variables are also Gaussian. It is easy to see that $E[2X + Y] = 0$, and
   \[ \text{Var}(2X + Y) = \text{Cov}(2X + Y, 2X + Y) = 4 \text{Var}(X) + 4 \text{Cov}(X,Y) + \text{Var}(Y) = 5 \]

   This shows that $2X + Y \sim N(0,5)$ and
   \[ P(2X + Y \geq 1) = P\left(\frac{2X + Y}{\sqrt{5}} \geq \frac{1}{\sqrt{5}}\right) = Q\left(\frac{1}{\sqrt{5}}\right). \]
3. [Gaussian Random variables and Estimation]

(a) Solution: Using the linearity of expectation and bi-linearity of covariance, we get

\[ \text{Var}(W) = \text{Cov}(2X + Y - 1, 2X + Y - 1) = 4 \text{Var}(X) + 4 \text{Cov}(X, Y) + \text{Var}(Y) = 37 \]

Here we use \( \text{Cov}(X, Y) = \rho \sigma_X \sigma_Y = 3 \).

(b) Solution: First, observe that \( W, Y \) are also jointly Gaussian since \( W \) is a linear combination of jointly Gaussian \( X, Y \). From the lecture, we know that the best mean square error estimator for jointly Gaussian random variables is the linear MSE, i.e.

\[ g^*(w_0) = E[Y] + \frac{\text{Cov}(W, Y)}{\text{Var}(W)} (w_0 - E[W]) \]

Since \( \text{Cov}(W, Y) = \text{Cov}(2X + Y - 1, Y) = 2 \text{Cov}(X, Y) + \text{Var}(Y) = 15 \), \( g^*(w_0) = -1 + \frac{15}{37} w_0 \), and the resulting MSE is

\[ E[(Y - g^*(W))^2] = \text{Var}[Y] \left( 1 - \rho^2_{W,Y} \right) = \frac{9 \cdot 12}{37} \]

4. [LLN]

Solution: Let \( Y = \frac{S_n}{n} - \mu \). It is easy to check that \( E[Y] = 0 \) and \( \text{Var}(Y) = \frac{\sigma^2}{n} \). Since \( Y \) is a linear combination of i.i.d. Gaussian R.V.’s, \( Y \) is also Gaussian, i.e. \( Y \sim N(0, \frac{\sigma^2}{n}) \). So we have

\[ P(|Y| \geq \epsilon) = P\left( \frac{|Y|}{\sigma/\sqrt{n}} \geq \frac{\epsilon}{\sigma/\sqrt{n}} \right) = 2Q\left( \frac{\epsilon/\sqrt{n}}{\sigma} \right) \]

Let \( X \sim N(0, 1) \), and for an interval \( I \) denote \( 1_I \) be the function defined as

\[ 1_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{otherwise} \end{cases} \]

To prove the limit,

\[ \lim_{n \to \infty} P \left( \left| \frac{S_n}{n} - \mu \right| \geq \epsilon \right) = \lim_{n \to \infty} Q\left( \frac{\epsilon/\sqrt{n}}{\sigma} \right) \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{\infty} f_X(x) \, dx \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{1\left(\frac{\epsilon/\sqrt{n}}{\sigma}\right)} 1 \, dx \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{1\left(\frac{\epsilon/\sqrt{n}}{\sigma}\right)} f_X(x) \, dx \]
\[ = \int_{-\infty}^{\infty} \lim_{n \to \infty} 1\left(\frac{\epsilon/\sqrt{n}}{\sigma}\right) f_X(x) \, dx \]
Since $\lim_{n \to \infty} \mathbb{1}_{(-\infty,x]} \left( \frac{\epsilon \sqrt{n}}{\sigma} \right) = 0$ for a fixed $x$, it follows

$$\lim_{n \to \infty} P \left( \left| \frac{S_n}{n} - \mu \right| \geq \epsilon \right) = \int_{-\infty}^{\infty} 0 \cdot f_X(x) \, dx = 0$$

Alternatively, we can use Chebyshev’s inequality to get the upper bound

$$P \left( \left| \frac{S_n}{n} - \mu \right| \geq \epsilon \right) \leq \frac{\sigma^2/n}{\epsilon^2}$$

Since $\epsilon, \sigma$ are fixed, taking the limit on the both sides proves $P \left( \left| \frac{S_n}{n} - \mu \right| \geq \epsilon \right) \to 0$ as $n \to \infty$. 