University of Illinois Fall 2022

ECE 313: Final Exam

Monday, Dec 12, 2022

1. [**5 points**] Recall the first problem in Fall 2022 midterm 2. Let X be a non-negative continuous random variable such that $P(X > x + y | X > y) = P(X > x)$ for all $x \geq 0$ and all $y \geq 0$. It was shown that $P(X > n) = (P(X > 1))^n$ for every positive integer n. Suppose that $P(X > 3) < \frac{1}{8}$. **True or False**: $P\left(X > \frac{1}{2}\right) < \frac{1}{\sqrt{2}}$. Justify your answer.

   Note: X is an exponential random variable. Nevertheless, this observation should not be used in your proof.

   Solution: By the midterm 2 solution, the memory-less property can be equivalently written as

$$P(X > x + y) = P(X > x)P(X > y), \quad \forall x \geq 0 \text{ and } \forall y \geq 0.$$

   For $x = y = \frac{1}{2}$, $\sqrt[3]{P(X > 3)} = P(X > 1) = (P\left(X > \frac{1}{2}\right))^2$ or $P\left(X > \frac{1}{2}\right) < \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$.

2. [**8 points**] Suppose that we have a five-sided die with equiprobable sides labeled $\underline{\{1, 2, 3, 4, 5\}}$ and a three-sided die with equiprobable sides labeled $\underline{\{1, 3, 5\}}$. We run the following experiment:

   - Roll 1: we roll the five-sided die.
   - Roll 2: if the result of the first roll is odd, then we roll the five-sided die. Otherwise, we roll the three-sided die.
   - Roll 3: if the sum of the first two rolls is odd, we roll the five-sided die. Otherwise, we roll the three-sided die.
   - Roll 4: if the sum of the first three rolls is odd, we roll the five-sided die. Otherwise, we roll the three-sided die.

   We get four numbers as the result of the above experiment.

   (a) [**4 points**] Let X denote the number of times we have rolled the three-sided die. Find the pmf of X.

   Solution: Note that we won't roll the three-sided die twice in a row; hence, possible values of X are $0, 1, 2$.

$$p_X(0) = P(X = 0) = \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^2 = \frac{12}{125}$$
$$p_X(2) = P(X = 2) = \left(\frac{2}{5}\right)(1)\left(\frac{3}{5}\right) = \frac{30}{125}$$
$$p_X(1) = P(X = 1) = 1 - \frac{12}{125} - \frac{30}{125} = \frac{83}{125}$$

   (b) [**4 points**] Let Y denote the number of times we have rolled the five-sided die. Find $E[X^3]$ and $P(X^2 \geq 1 | Y \geq 1)$.

   Solution:
   $E[X^3] = 0^3 \cdot p_x(0) + 1^3 \cdot p_x(1) + 2^3 \cdot p_x(2) = \frac{323}{125}$
   $P(X^2 \geq 1 | Y \geq 1) = P(X \geq 1) = 1 - \frac{12}{125} = \frac{113}{125}$

3. **[7 points]** Let X, Y be two non-negative random variables such that $\mu_X = E[X] = E[Y] = \mu_Y = 2$, $\sigma_X^2 = \text{Var}(X) = \sigma_Y^2 = \text{Var}(Y) = 1$. Using Markov, Chebyshev and Cauchy-Schwarz inequalities, justify your answer for the following:

   (a) [**4 points**] **True or False**: $P(XY \geq 10) \leq \frac{1}{2}$.

   Solution: $P(XY \geq 10) \leq \frac{E[XY]}{10} \leq \frac{\sqrt{E[X^2]E[Y^2]}}{10} = \frac{\sqrt{(\mu_X^2+\sigma_X^2)(\mu_Y^2+\sigma_Y^2)}}{10} = \frac{1}{2}$.

   (b) [**3 points**] **True or False**: $P(X \geq 6) \leq \frac{1}{16}$.

   Solution: $P(X \geq 6) = P(X - 2 \geq 4) \leq P(|X - 2| \geq 4) \leq \frac{\sigma_X^2}{4^2} = \frac{1}{16}$.

4. **[7 points]** Let $N \sim \text{Pois}(\lambda)$ and given N, consider N independent Bernoulli trials, each with probability of success p. Let X be the number of successes in these N trials.

   (a) [**4 points**] Find $P(N = n|X = k)$ for $n \geq k$.

   Solution: $P(N = n|X = k) = \frac{P(X=k|N=n)P(N=n)}{P(X=k)} = \frac{\binom{n}{k}p^k(1-p)^{n-k}e^{-\lambda}\frac{\lambda^n}{n!}}{\sum_{j\geq k}\binom{j}{k}p^k(1-p)^{j-k}e^{-\lambda}\frac{\lambda^j}{j!}} = e^{-\lambda(1-p)}\frac{[\lambda(1-p)]^{n-k}}{(n-k)!}$.

   (b) [**3 points**] Find the best unconstrained estimator of N given $X = k$.

   Solution: $E[N|X = k] = \sum_{n\geq k} ne^{-\lambda(1-p)}\frac{[\lambda(1-p)]^{n-k}}{(n-k)!} = \sum_{n\geq k}(n-k+k)e^{-\lambda(1-p)}\frac{[\lambda(1-p)]^{n-k}}{(n-k)!} = k + \lambda(1 - p)$.

5. **[12 points]** Suppose that the pdf of a random variable X is given by

$$f(x) = \begin{cases} ax + a^2 & \text{if } x \in [0, |a|\,] \\ 0 & \text{o.w.} \end{cases}$$

   for some real number a.

   (a) [**4 points**] Find $a > 0$ and $a < 0$ such that the resulting pdf is valid.

   Solution: Note that for all $x \in [0, |a|\,]$, the value of $ax + a^2$ is positive, hence, it is enough to check $\int_0^{|a|} f(x)dx = 1$.

$$\int_0^{|a|} f(x)dx = \int_0^{|a|} (ax + a^2)dx = a \times \frac{|a|^2}{2} + a^2 \times |a| = 1$$

   If $a > 0$, then the solution of the above equation is $a = \sqrt[3]{\frac{2}{3}}$, and if $a < 0$, then the solution of the above equation is $a = -\sqrt[3]{2}$.

   (b) [**8 points**] For the obtained values of a, let $H_0 : a > 0$, i.e., $f_0(x) = f(x)$ for $a > 0$ and $H_1 : a < 0$, i.e, $f_1(x) = f(x)$ for $a < 0$. Given an observation $X = u$, describe the ML decision rule.

   Solution: Based on the previous part, the hypothesis H_0 and H_1 is given as follows:

   - H_0: $a = \sqrt[3]{\frac{2}{3}}$, i.e., the pdf of X is given by

$$f_0(x) = \begin{cases} \sqrt[3]{\frac{2}{3}}x + \sqrt[3]{\frac{4}{9}} & \text{if } x \in [0, \sqrt[3]{\frac{2}{3}}\,] \\ 0 & \text{o.w.} \end{cases}$$

   - H_1: $a = -\sqrt[3]{2}$, i.e., the pdf of X is given by

$$f_1(x) = \begin{cases} -\sqrt[3]{2}x + \sqrt[3]{4} & \text{if } x \in [0, \sqrt[3]{2}\,] \\ 0 & \text{o.w.} \end{cases}$$

The likelihood ratio for an observation $X = u$ is given by

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} \tag{1}$$

Hence, the ML decision rule is

$$\begin{cases} H_1 & \text{if } u \in [0, \sqrt[3]{2} - \sqrt[3]{\frac{2}{3}}\,] \cap [\sqrt[3]{\frac{2}{3}}, \sqrt[3]{2}\,] \\ H_0 & \text{o.w.} \end{cases} \tag{2}$$

Note that if $f_1(u) = f_0(u)$, then

$$u = \frac{\sqrt[3]{4} - \sqrt[3]{\frac{4}{9}}}{\sqrt[3]{2} + \sqrt[3]{\frac{2}{3}}} = \sqrt[3]{2} - \sqrt[3]{\frac{2}{3}} \tag{3}$$

6. **[8 points]** Suppose X and Y are independent random variables such that X is uniformly distributed over $[0, 1]$ and Y is uniformly distributed over $[-1, 0]$. Let $W = 2X - Y$ and $Z = X + Y$. Find the joint pdfs f_{XY} and f_{WZ}.

   Solution: Since X and Y are independent random variables,

   $$f_{X,Y}(u, v) = \begin{cases} 1, & u \in [0, 1] \text{ and } v \in [-1, 0], \\ 0, & \text{else.} \end{cases}$$

   Also,

   $$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

   We apply Proposition 4.7.1 in lecture notes, $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det A|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$, using

   $$\det(A) = 3 \quad A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

   Specifically, the random variables W and Z have joint pdf given by

   $$\begin{aligned} f_{W,Z}(\alpha, \beta) &= \frac{1}{3} f_{X,Y}\left(\frac{1}{3}\alpha + \frac{1}{3}\beta, -\frac{1}{3}\alpha + \frac{2}{3}\beta\right) \\ &= \begin{cases} \frac{1}{3}, & \alpha \in [2\beta, 3 - \beta] \text{ and } \beta \in [0, 1] \\ \frac{1}{3}, & \alpha \in [-\beta, 3 + 2\beta] \text{ and } \beta \in [-1, 0] \\ 0, & \text{else.} \end{cases} \end{aligned}$$

7. **[9 points]** Six balls numbered one through six are in a bag. Two balls are drawn at random, without replacement, with all possible outcomes having equal probability. Let X be the number on the one ball drawn and Y be the number on the other ball drawn.

   (a) **[3 points]** Find $E[X]$ and $\text{Var}(X)$.

   Solution: The joint pmf for X and Y is $p_{X,Y}(k_1, k_2) = \frac{1}{30}$ for $k_1, k_2 \in \{1, 2, \ldots, 6\}$ and $k_1 \neq k_2$. The pmf for X is $p_X(k) = \sum_{k_2} p_{X,Y}(k, k_2) = \frac{1}{6}$ for all $k = 1, 2, \ldots, 6$. Therefore,

   $$E[X] = \frac{1}{6}\sum_{k=1}^{6} k = \frac{7}{2},$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{6}\sum_{k=1}^{6} k^2 - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

(b) [**3 points**] Find $E[Y]$ and $\text{Var}(Y)$.
Solution: The pmf for Y is $p_Y(k) = \sum_{k_1} p_{X,Y}(k_1, k) = \frac{1}{6}$ for all $k = 1, 2, \ldots, 6$,

$$E[Y] = \frac{1}{6}\sum_{k=1}^{6} k = \frac{7}{2},$$

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{1}{6}\sum_{k=1}^{6} k^2 - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

(c) [**3 points**] Find the correlation coefficient $\rho_{X,Y}$. Are X and Y independent?
Solution:

$$E[XY] = \frac{1}{6 \times 5}\left(\sum_{i=1}^{6}\sum_{j=1}^{6} ij - \sum_{i=1}^{6} i^2\right) = \frac{1}{30}(21 \times 21 - 91) = \frac{350}{30} = \frac{35}{3}.$$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{35}{3} - \frac{49}{4} = -\frac{7}{12}.$$

The correlation coefficient is $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = -\frac{7/12}{35/12} = -\frac{7}{35}$. X and Y are not independent since the correlation coefficient is non-zero.

8. [**12 points**] Suppose X and Y are jointly Gaussian random variables with $E[X] = 2$, $E[Y] = 0$, $\text{Var}(X) = 16$, $\text{Var}(Y) = 4$, and $\rho = 0.25$. Let $Z = X + 2Y + 1$.

(a) [**4 points**] Find $E[Z]$, $\text{Var}(Z)$, and the pdf of Z.
Solution:

$$E[Z] = E[X + 2Y + 1] = 3E[X] + E[Y] + 1 = 2 + 1 = 3.$$

The covariance between X and Y, $\text{Cov}(X, Y) = \rho\sqrt{\text{Var}(X)\text{Var}(Y)} = 0.25 \times 4 \times 2 = 2.$

$$\begin{aligned}
\text{Var}(Z) &= \text{Var}(X + 2Y) \\
&= \text{Cov}(X + 2Y, X + 2Y) \\
&= \text{Var}(X) + 4\text{Var}(Y) + 4\text{Cov}(X, Y) \\
&= 16 + 16 + 8 \\
&= 40
\end{aligned}$$

Since X and Y are jointly Gaussian random variables, $Z = X + 2Y + 1$ should be a Gaussian random variable as well, and $Z \sim \mathcal{N}(3, 40)$. The pdf of Z should be

$$\begin{aligned}
f_Z(z) &= \frac{1}{\sqrt{2\pi}\sqrt{40}} e^{-\frac{1}{2}\left(\frac{z-3}{\sqrt{40}}\right)^2} \\
&= \frac{1}{\sqrt{80\pi}} e^{-\frac{(z-3)^2}{80}}.
\end{aligned}$$

(b) [**4 points**] Find $E[Y|Z]$.

Solution: Since X and Y are jointly Gaussian and Z is a linear combination of X and Y, $E[Y|Z] = \hat{E}[Y|Z] = \mu_Y + \frac{\text{Cov}(Y,Z)}{\text{Var}(Z)}(Z - \mu_Z)$. The covariance between Y and Z is $\text{Cov}(Y,Z) = \text{Cov}(Y, 3X+Y+1) = 3\text{Cov}(Y,X) + \text{Var}(Y) = 3 \times 2 + 4 = 10$. Therefore,

$$E[Y|Z] = 0 + \frac{10}{40}(Z-3) = \frac{1}{4}Z - \frac{3}{4}.$$

(c) [**4 points**] Find the mean square error, $E\left[(Y - E[Y|Z])^2\right]$

Solution: Since Y and Z are jointly Gaussian, the mean square error satisfies,

$$E\left[(Y - E[Y|Z])^2\right] = E\left[(Y - \hat{E}[Y|Z])^2\right] = \text{Var}(Y)(1 - \rho_{Z,Y}^2) = \frac{3}{2}$$

9. [**32 points**] (2 points per answer)
In order to discourage guessing, 2 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose A, B and C are three events with positive probabilities.

TRUE	FALSE	
☐	☐	If A and B are independent, then they are mutually exclusive.
☐	☐	If A, B and C are pairwise independent events, then A, B and C are mutually independent.
☐	☐	If $P(A) = 0.7$ and $P(B) = 0.4$, then $0.1 \leq P(AB) \leq 0.4$.
☐	☐	$P(A\|C) = P(B)P(A\|B,C) + P(B^c)P(A\|B^c,C)$.
☐	☐	$P(A\|B) \leq P(A)$.

Solution: False, False, True, False, False

(b) Consider two random variables X and Y.

TRUE	FALSE	
☐	☐	The maximum possible value of $\text{Cov}(X,Y)$ is 4 if $\text{Var}(X) = 8$ and $\text{Var}(Y) = 2$.
☐	☐	If $E[X] = E[Y] = 1$, $E[X^2] = 2$, $E[Y^2] = 4$ and $E[XY] = 2$, then $\text{Var}(X+Y) = 6$.
☐	☐	If X and Y are independent and uniformly distributed on $[0,1]$, then $X+Y$ is uniformly distributed on $[0,2]$
☐	☐	If X has Poisson distribution, then, $E[X] < \text{Var}(X)$.
☐	☐	If X has geometric distribution, then $P(X > k+n\|X > n) = P(X > k)$ for all positive integers k and n.

Solution: True, True, False, False, True

(c) Consider two jointly Gaussian random variables X and Y that are uncorrelated.

TRUE	FALSE	
☐	☐	The MMSE for estimating X by observing Y is $\text{Var}(X)$.
☐	☐	If $\mu_X = \mu_Y = 1$ and $\sigma_X^2 = \sigma_Y^2 = 2$, then $E[XY + 2X + Y + 1] = 4$.

Solution: True, False

(d) Consider a hypothesis testing problem.

TRUE	FALSE	
☐	☐	Always $p_{\text{miss,MAP}} < p_{\text{miss,ML}}$.
☐	☐	If $\pi_1 = \pi_0 = 1/2$, then the ML and the MAP estimators are the same.

Solution: False, True

(e) Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of random variables with mean $-\infty < \mu < \infty$ and bounded variance. Suppose that $S_n = X_1 + X_2 + \ldots + X_n$.

TRUE	FALSE	
☐	☐	If X_1 is initially sampled and $X_2 = X_3 = \cdots = X_n = X_1$, then, by the LLN, $P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0$ as $n \to \infty$ for any $\epsilon > 0$.
☐	☐	If $\text{Cov}(X_i, X_j) = 0$ for all i and j with $i \neq j$, then $P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0$ as $n \to \infty$ for any $\epsilon > 0$.

Solution: False, True