Lecture 6 - 9/2

Review:

① Mean, Variance, Moments

(1.1) mean, average, expected value: $E[X] = \sum_i u_i P_X(u_i)$

LOTUS: $E[g(X)] = \sum_i g(u_i) P_X(u_i) = \mu_X$

(1.2) Variance & standard deviation:

Variance: $Var(X) = E\left[(x-\mu_X)^2\right] = \sum_i (u_i - \mu_X)^2 P_X(u_i)$

$= E[X^2] - \mu_X^2$

Standard deviation: $\sigma_X = \sqrt{Var(X)}$, $\sigma_X^2 = Var(X)$

Standardized version: $Y = \frac{X-\mu_X}{\sigma_X}$, has no unit, $E[Y]=0$, $Var(Y)=1$

(1.3) Properties: follows by LOTUS

$Var(aX) = a^2 \sigma_X^2$, $\sigma_{aX} = |a| \sigma_X$

$E[aX] = a\mu_X$, $E[X+b] = \mu_X + b$

$Var(X+b) = Var(X)$

(1.4) i'th moment $= E[X^i]$, $i \geq 1$ integer

② Conditional probability:

- $P(B|A) = \begin{cases} \frac{P(AB)}{P(A)} & P(A) > 0 \\ \text{undefined} & P(A) = 0 \end{cases}$

Given the information that A happened, how likely is B?

- Frequency interpretation

We run an experiment N times.

N_A = # of times A happened

N_A = # of times A happened

N_{AB} = # of times AB happened

$P(B|A) \approx \frac{N_{AB}}{N_A}$

. $P(\cdot|A)$ is a probability measure, satisfies axioms of probability. Given $P(A)>0$,

① $P(\Omega|A)=1$

② $P(B|A)\geq 0$ for all $B\in F$

③ If $B_1, B_2, \ldots$ are mutually exclusive $P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i)$

Today:

① Bayes rule

② Independence

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① Bayes rule: $P(B|A) \cdot P(A) = P(A|B)P(B) = P(AB)$ given $P(A), P(B)>0$

in particular $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

② Independence:

Def: A & B are independent if $P(AB) = P(A)P(B)$.

Intuition: suppose that $P(A)>0$. Then A & B are independent if knowledge of A has no implication on B, i.e.,

$$P(B|A) = P(B)$$

note: $P(AB) = P(A)P(B)$ is just an equation:

* roll a die, $A=\{2,4,6\}$. $B=\{3,6\}$, $AB=\{6\}$

A and B are independent: $P(AB) = P(\{6\}) = \frac{1}{6} = P(A)P(B)$

Def: A, B, C are pairwise independent if

$$P(AB) = P(A)P(B)$$

$$P(AC) = P(A)P(C)$$

$$P(BC) = P(B)P(C)$$

Def: A, B, C are independent if they are pairwise independent and

$$P(ABC) = P(A)P(B)P(C)$$

Def: $A_1, A_2, \ldots, A_n$ are independent if

$$P(A_{i_1} A_{i_2} \cdots A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

for any $2 \leq k \leq n$ & $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$

note: A, B, C being pairwise independent won't imply independence.

* roll a 9-Sided die. Sides are $\{1, 2, \ldots, 9\}$. Outcomes are equilikely.

$$A = \{1, 2, 3\} \quad B = \{3, 4, 5\} \quad C = \{2, 4, 6\}$$

$$P(A) = P(B) = P(C) = \frac{3}{9}$$

$$P(AB) = P(BC) = P(AC) = \frac{1}{9}$$

$$P(ABC) = 0$$

. Implications:

- A independent of B, then A^c is independent of B.

$$P(B) = P(B \cap (A \cup A^c)) = P(B \cap A) + P(B \cap A^c)$$

$$\rightarrow P(A^c \cap B) = (1 - P(A))P(B) = P(A^c)P(B)$$

- A, B, C are independent, then A is independent of $B \cup C$

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$

$$= P(A\cap B) + P(A\cap C) - P(A\cap B\cap C)$$

$$= P(A)\,(P(B) + P(C) - P(BC)$$

$$= P(A)P(B\cup C)$$

In particular, A is independent of any combination of B & C.

- Above properties generalizes to $A_1, A_2, \dots, A_n$.

  B_1: combination of n_1 out of $\{A_1, \dots, A_n\}$

  B_2: combination of n_2 out of $\{A_1, \dots, A_n\}$ that did not appear in B_1

  $\vdots$

  B_K: combinations of n_K out of $\{A_1, \dots, A_n\}$ that did not appear in $B_1, \dots, B_{K-1}$

Then $B_1, B_2, \dots, B_K$ are independent.

Important remarks:

① independence vs mutually exclusive:

- We say A & B are independent, if $P(AB) = P(A)P(B)$
- We say A & B are mutually exclusive, if $A\cap B = \emptyset$

In particular, being independent relies on the underlying probability measure P; being mutually exclusive has nothing to do with probability measure and it is a set level definition.

② intersection vs conditional probability.

- $A\cap B$: the event A AND B. It is a set level definition.
- $P(B|A)$: the probability of event B GIVEN that A has happened. It depends on a probability measure P.

Here is a semi-example:

Suppose that Jane lost his dog, and the dog is in forest A or forest B. Jane uses a coin to decide which forest to search.

- What is the probability that Jane will search forest A & find the dog?

  the question is asking for

  $$P(\{\text{Jane will search forest A}\} \cap \{\text{dog is in A and search is successful}\})$$

- After tossing the coin, Jane decides to search forest A. What is the probability that Jane will find the dog?

  the question is asking for

  $$P(\{\text{dog is in A and search is successful}\} \mid \{\text{Jane will search forest A}\})$$