Week 11 and 12.

- Independence of random variables:
  - If $X$ & $Y$ are independent if & only if $F_{X,Y}(uv) = F_X(u)F_Y(v)$ $\forall u,v \in \mathbb{R}$.  
  - If $X$ & $Y$ are discrete type, they are independent if & only if $P_{X,Y}(uv) = P_X(u)P_Y(v)$ $\forall u,v \in \mathbb{R}$.  
  - If $X$ & $Y$ are jointly continuous, they are independent if & only if $f_{X,Y}(uv) = f_X(u)f_Y(v)$ $\forall u,v \in \mathbb{R}$.

- Determining from a joint pdf whether independence holds

To ensure independence, for every $u \in \mathbb{R}$, either $f_{X,Y}(uv) = f_X(u)$ only depends on $u$, i.e., $f_{Y|X}(v|u) = f_Y(v)$.

To check for independence, if $X$ & $Y$ are independent then support of $f_{X,Y}$ should satisfy swap property, i.e.,

$$(a,b) \in \text{support of } f_{X,Y} \quad (a,d) \in \text{support of } f_{X,Y} \quad \downarrow$$

$$(c,d) \in \text{support of } f_{X,Y} \quad (b,c) \in \text{support of } f_{X,Y} \quad \uparrow$$

- Function of pair of jointly continuous random variables.

$Z = g(X,Y)$ & the joint pdf of $(X,Y)$ is given by $f_{X,Y}$. Find distribution of $Z$:

**Step 1:** Identify type of $Z$ (continuous type or discrete type) and support of $Z$ (c st. $f_Z(z) > 0$)

- If $Z$ is continuous type, 
  - Step 2: Find its cdf.
  - $F_Z(c) = P(Z \leq c) = P(g(X,Y) \leq c) = \int \int f_{X,Y}(uv) \, du \, dv$.

- If $Z$ is discrete type, 
  - Step 2: Calculate pmf of $Z$.
  - $P(Z = k) = P(g(X,Y) = k) = \int \int f_{X,Y}(uv) \, du \, dv$.

- Sum of random variables:

  Suppose that $X$ & $Y$ are discrete type, and integer valued. Let $S = X + Y$:

  $$P(S = s) = \sum_{(i,j) \in I} P(Y = j | X = i) P(X = i)$$

  $$P(S = s) = \sum_{(i,j) \in I} P(Y = j | X = i) P(X = i)$$
\[ P_S(k) = P(S = k) = \sum_j P(X = j, Y = k - j) = \sum_j P_{XY}(j, k - j) \]

If \( X \) and \( Y \) are independent, then

\[ P_S(k) = P(S = k) = \sum_j P_X(j) P_Y(k - j) =: P_X \ast P_Y(k) \text{ if } X \text{ and } Y \text{ are independent & } S = X + Y \]

Suppose that \( X \) and \( Y \) are jointly continuous. Let \( S = X + Y \):

\[ f_S(c) = \int_{-\infty}^{\infty} f_{XY}(u, c-u) \, du = \int_{-\infty}^{\infty} f_{XY}(c-v, v) \, dv \]

If \( X \) and \( Y \) are independent,

\[ f_S(c) = \int_{-\infty}^{\infty} f_X(u) f_Y(c-u) \, du = \int_{-\infty}^{\infty} f_X(c-v) f_Y(v) \, dv = f_X \ast f_Y(c) \text{ if } X \text{ and } Y \text{ are independent & } S = X + Y \]

**Week 13:**

Joint pdf of functions of random variables.

Suppose that \( X \) and \( Y \) are jointly continuous random variables with joint pdf \( f_{XY} \).

Suppose that \( U = g_1(X, Y) \) and \( Z = g_2(X, Y) \). Define \( g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as follows.

We can write:

\[ g((x, y)) = (\frac{g_1(x, y)}{g_2(x, y)}) \]

\[ \begin{pmatrix} W \\ Z \end{pmatrix} = g \left( \begin{pmatrix} X \\ Y \end{pmatrix} \right) \]

Suppose that \((X, Y)\) is in \( u-v \) plane & \((W, Z)\) is in \( u-\beta \) plane.

Let \( J(u, v) \) denote the Jacobian matrix of \( g \) at point \((u, v)\).

\[ J(u, v) = \begin{bmatrix} \frac{\partial g_1(u, v)}{\partial u} & \frac{\partial g_1(u, v)}{\partial v} \\ \frac{\partial g_2(u, v)}{\partial u} & \frac{\partial g_2(u, v)}{\partial v} \end{bmatrix} \]
Proposition: Suppose that \((W, Z) = g(X, Y)\), where \((X, Y)\) has pdf \(f_{X,Y}\), and \(g\) is one to one mapping from support of \(f_{X,Y}\) to \(\mathbb{R}^2\). Suppose that the Jacobian matrix \(J\) of \(g\) exists, is continuous, and has nonzero determinant everywhere. Then for all \((w, z)\) in the support of \(f_{W,Z}\) we have

\[
f_{W,Z}(w, z) = \frac{1}{|\det(J)|} f_{X,Y}(g^{-1}(w, z))
\]

Important remark: The Jacobian matrix \(J\) depends on \(w\) and \(v\), i.e., \(J = J(wv)\); however \(f_{W,Z}(w, z)\) is in terms of \(w\) and \(z\).

You should calculate \(J\) in terms of \(w\) and \(z\), and then write it in terms of \(w\) and \(z\).

Example:

\[
\begin{bmatrix}
w \\
z
\end{bmatrix} = \begin{bmatrix}
A(X) \\
Y
\end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

\(A\) is invertible

\[
f_{W,Z}(w, z) = \frac{1}{|\det(A)|} f_{X,Y}(A^{-1}(w, z))
\]

Correlation, Covariance, Correlation Coefficient.

Let \(X\) and \(Y\) be random variables with finite second moments.

- The correlation, \(\text{E}[XY]\)
- The covariance, \(\text{Cov}(X, Y) = \text{E}[(X - \text{E}[X])(Y - \text{E}[Y])]\)
- The correlation coefficient: \(\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}\)

- Scaling and linearity of Covariance.
  \(i\) \(\text{Cov}(X+Y, U+V) = \text{Cov}(X, U) + \text{Cov}(X, V) + \text{Cov}(Y, U) + \text{Cov}(Y, V)\)
  \(ii\) \(\text{Cov}(ax + c, by + d) = ac \text{Cov}(X, Y)\)

Important def. & relations.

- Unrelated, negatively correlated, positively correlated
- \(\text{Var}(X) = \text{Cov}(X, X)\). In particular if \(X\) and \(Y\) are unrelated
  \(\text{Var}(X+Y) = \text{Cov}(X+Y, X+Y) = \text{Cov}(X, X) + \text{Cov}(Y, Y) = \text{Var}(X) + \text{Var}(Y)\)

- If \(X\) & \(Y\) are independent, they are uncorrelated. The reverse does not hold in general.
  (Note: If \(X\) & \(Y\) are jointly Gaussian & uncorrelated, they are independent.)
If \( X \) and \( Y \) are independent, they are uncorrelated. The converse does not hold in general.

(Note: If \( X \) and \( Y \) are jointly Gaussian and uncorrelated, they are independent.)

**Week 14 & 15.**

**Estimation & Minimum Mean Square Error (MMSE).**

1. **Sample mean & variance of data set**

   Given a independent & identically distributed random variables \( X_1, X_2, X_3, \ldots, X_n \), the sample mean & variance are unbiased estimator and are given as follows:
   
   \[
   \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\mu})^2
   \]

2. **Minimum mean square error estimators of random variable \( Y \), after observing random variable \( X \)**

   - **Constant estimator:** \( \hat{Y} = E[Y] \), \( \text{MMSE} = \sigma_Y^2 \)
   - **Linear estimator:** \( \hat{Y}(X) = \hat{\beta}_Y + \hat{\gamma}_Y \mu_X \) \( \hat{\beta}_Y = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \), \( \text{MMSE} = \sigma_Y^2 (1 - \hat{\beta}_Y^2) \)
   - **Unconstrained estimator:** \( \hat{\gamma}(X) = E[Y|X] \), \( \text{MMSE} = \sigma_Y^2 - E[(E[Y|X])^2] \)

**Limit Theorems.**

3. **Law of large numbers**

   Suppose that \( X_1, X_2, \ldots, X_n \) are uncorrelated, each one has the same mean \( \mu \), and their variance is bounded by \( C \). Let \( S_n = X_1 + X_2 + \cdots + X_n \). Then, for any \( s > 0 \), we have
   
   \[
P(\{|S_n - \mu| \geq s\}) = P(\{|\text{Var}(X)\phi_n(S_n - \mu)| \geq s\}) \leq \frac{C}{n^2} \quad \text{as} \quad n \to \infty.
   \]

4. **Central limit theorem**

   Suppose that \( X_1, X_2, \ldots, X_n \) are independent, identically distributed, each with mean \( \mu \) and variance \( \sigma^2 \). Let \( S_n = X_1 + X_2 + \cdots + X_n \). Then, for any \( c \), we have
   
   \[
   \lim_{n \to \infty} P\left( \left\{ \frac{S_n - \mu}{\sigma \sqrt{n}} \leq c \right\} \right) = \lim_{n \to \infty} P\left( \left\{ \frac{\hat{S}_n - \mu}{\sigma \sqrt{n}} \leq c \right\} \right) = \Phi(c)
   \]

   where \( S_n = \frac{S_n - \mu}{\sigma \sqrt{n}} = \frac{S_n - E(S_n)}{\sqrt{\text{Var}(S_n)}} \) is the standardized version of \( S_n \).

**Jointly Gaussian random variables.**

We say random variables \( X \) and \( Y \) are jointly Gaussian random variables if any linear combination of \( X \) and \( Y \) are Gaussian random variables, i.e., for any \( a, b \in \mathbb{R} \), \( aX + bY \) is a Gaussian random variables.
we say random variables \( X \) and \( Y \) are jointly Gaussian random variables if any linear combination of 
\( X \) and \( Y \) are Gaussian random variables, i.e., for any \( a, b \in \mathbb{R} \), \( aX + bY \) is a Gaussian random variables.

1. joint distribution of Gaussian random variables. Suppose that \( X, Y \) are non-degenerate, i.e., they are not linearly related. Then their joint distribution is given by:

\[
f_{X,Y}(u,v) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp \left( -\frac{\left( \frac{u-\mu_X}{\sigma_X} \right)^2 + \left( \frac{v-\mu_Y}{\sigma_Y} \right)^2 - 2\rho \left( \frac{u-\mu_X}{\sigma_X} \right) \left( \frac{v-\mu_Y}{\sigma_Y} \right)}{2(1-\rho^2)} \right)
\]

where,

\[
\mu_X = \mathbb{E}[X], \quad \sigma_X^2 = \text{Var}(X)
\]

\[
\mu_Y = \mathbb{E}[Y], \quad \sigma_Y^2 = \text{Var}(Y)
\]

\[
\rho = \rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}
\]

This is called bivariate normal distribution.

2. class of bivariate normal pdf’s are preserved via linear transformation.

\[
\begin{pmatrix} X \\ Y \end{pmatrix}
\]

has bivariate distribution \( \Rightarrow \)

\[
\begin{pmatrix} X \\ Y \end{pmatrix}
\]

has bivariate distribution

This same as in definition of jointly Gaussian random variables.

\[
\begin{array}{c}
X \text{ & } Y \text{ are jointly Gaussian} \Rightarrow \text{ } aX + bY \text{ & } cX + dY \text{ are jointly Gaussian}
\end{array}
\]

3. If \( \rho = 0 \) then \( f_{X,Y}(u,v) = f_X(u)f_Y(v) \). Hence uncorrelated jointly Gaussian random variables are independent.

4. If \( X \) and \( Y \) are independent, then \( \rho = 0 \) and their joint distribution \( f_{X,Y}(u,v) = f_X(u)f_Y(v) \) is bivariate normal distribution. Hence, independent Gaussian random variables are jointly Gaussian.

5. If \( X \) and \( Y \) are Gaussian, it does not mean that they are jointly Gaussian.

6. If \( X \) and \( Y \) are jointly Gaussian, then \( X \) has \( N(\mu_X, \sigma_X^2) \) distribution and \( Y \) has \( N(\mu_Y, \sigma_Y^2) \) distribution.

7. If \( X \) and \( Y \) are jointly Gaussian, for estimation of \( Y \) from \( X \), \( \hat{f}(x) = \hat{g}(x) \). Equivalently, \( \mathbb{E}[Y|X] = \hat{E}[Y|X] \)

\[
\hat{g}(x) = \mathbb{E}[Y|X] = \hat{E}[Y|X] = \mu_Y + \sigma_Y \rho_{X,Y} \left( \frac{X-\mu_X}{\sigma_X} \right)
\]

8. If \( X \) and \( Y \) are jointly Gaussian, the conditional distribution of \( Y \) given \( X = u \) is \( N(\hat{E}[Y|X=u], \sigma_Y^2) \).
If $X$ and $Y$ are jointly Gaussian, the conditional distribution of $Y$ given $X = u$ is $N(\hat{E}[Y|X = u], \sigma_Y^2)$ where $\sigma_Y^2$ is MSE for $\hat{E}[Y|X]$.

$$\sigma_Y^2 = \sigma_Y^2 \left(1 - \frac{\sigma_X^2}{\sigma_{XY}^2}\right)$$

10. [30 points] (3 points per answer)

In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose $X$ and $Y$ are independent, continuous-type random variables. $W = \max(X, Y), Z = \min(X, Y)$.

TRUE    FALSE
☐    ☐  $F_W(t) = F_X(t)F_Y(t)$

☐    ☐  $F_Z(t) = (1 - F_X(t))(1 - F_Y(t))$

☐    ☐  $f_Z(t) = f_X(t)(1 - F_Y(t)) + f_Y(t)(1 - F_X(t))$

(b) Let $A, B, C$ be independent events with $0 < P(A), P(B), P(C) < 1$.

TRUE    FALSE
☐    ☐  $P(AC|B) = P(AC|B^c)$

☐    ☐  $P(AB|C) \leq P(C|AB)P(AB)$

(c) Let $X, Y$ be two independent normal random variables with $\mu_X = \mu_Y$ and with $\sigma_X > \sigma_Y$.

TRUE    FALSE
☐    ☐  $P(\leq X) > 1/2.$

☐    ☐  $P(X = Y) > 0$

(d) $X$ and $Y$ are jointly distributed discrete random variables. They are uncorrelated if:

TRUE    FALSE
☐    ☐  $\text{Var}(X + Y) = \text{Var}(X - Y)$

☐    ☐  $E[XY] = 0$

☐    ☐  $P(X = u, Y = v) = P(X = u)P(Y = v)$ for every pair $(u, v)$
7. [7+16+7 points] Suppose $X \sim N(1, 1)$ and $Y \sim N(1, 4)$ are independent Gaussian random variables. Define the random variables $Z = 2X + Y$ and $W = X - Y$.

(a) Find the unconstrained MMSE estimator of $Y$ given $X$, and the resulting MSE.

(b) Find the unconstrained MMSE estimator of $Z$ given $W$, and the resulting MSE.

(c) If instead $W = X - aY$ for some real $a$ and $E[Z|W] = E[Z]$, find $a$.

10. [10+10 points] Suppose $X$ and $Y$ are jointly Gaussian with the following parameters: $\mu_x = 0$, $\mu_y = 0$, $\sigma_x^2 = 1$, $\sigma_y^2 = 2^2$, $\rho = 1/8$.

(a) Find $P\{2X + Y \geq 3\}$. Express your answer using the Q function.

(b) Find $E[Y^2|X = 2]$

7. [15 points] The two parts of the problem are unrelated.

(a) Suppose a fair die is rolled 100 times. What is a rough approximation to the sum of the numbers showing, based on the law of large numbers?

(b) Suppose each of 1200 real numbers are rounded to the nearest integer and then added. Assume the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. The random variable equal to the sum is denoted by $S$. Using the CLT, find the approximate probability that the absolute value of the sum of the errors is greater than 5.

3. [8+4 points] Suppose $X$ and $Y$ are independent random variables with joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} 2e^{-u}e^{-2v} & \text{if } u \geq 0, v \geq 0 \\ 0 & \text{else.} \end{cases}$$

(a) Find the joint pdf of $S = X + Y$ and $W = Y - X$.

(b) Are $S$ and $W$ independent? Explain.