Review Lecture

Wednesday, December 07, 2022 10:30 AM

Week 11 and 12.

. Independence of random variables:

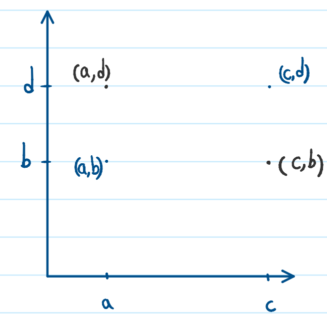
- X & Y are independent if & only if $F_{X,Y}(u,v) = F_X(u)F_Y(v) \quad \forall u,v \in \mathbb{R}$
- If X & Y are discrete type, they are independent if & only if $p_{X,Y}(u,v) = p_X(u)p_Y(v) \quad \forall u,v \in \mathbb{R}$
- If X & Y are jointly continuous, they are independent if & only if $f_{X,Y}(u,v) = f_X(u)f_Y(v) \quad \forall u,v \in \mathbb{R}$

. Determining from a joint pdf whether independence holds

To ensure independence: for every $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$ only depends of v, i.e., $f_{Y|X}(v|u) = f_Y(v)$.

To roll-out independence: if X & Y are independent then support of $f_{X,Y}$ should satisfy swap property, i.e.,

$$\begin{matrix} (a,b) \in \text{support of } f_{X,Y} \\ (c,d) \in \text{support of } f_{X,Y} \end{matrix} \Rightarrow \begin{matrix} (a,d) \in \text{support of } f_{X,Y} \\ (b,c) \in \text{support of } f_{X,Y} \end{matrix}$$



. Function of pair of jointly continuous random variables:

$Z = g(X,Y)$ & the joint pdf of (X,Y) is given by $f_{X,Y}$. Find distribution of Z.

Step 1: identify type of Z (continuous-type or discrete-type) and support of Z (c s.t. $f_Z(c) > 0$)

→ if Z is continuous-type:

Step 2: find its cdf. $F_Z(c) = P(Z \leq c) = P(g(X,Y) \leq c) = \iint_{(u,v):\, g(u,v) \leq c} f_{X,Y}(u,v)\, du\, dv$

Step 3: take derivative of $F_Z(c)$ to derive pdf of Z:

$$f_Z(c) = \frac{\delta F_Z(c)}{\delta c}$$

→ if Z is discrete-type: Step 2: calculate pmf of Z. $P(Z = k) = P(g(X,Y) = k) = \iint_{(u,v):\, g(u,v) = k} f_{X,Y}(u,v)\, du\, dv$

. Sum of random variables:

① Suppose that X & Y are discrete-type, and integer valued. Let $S = X + Y$.

$P_S(k) = P(S = k) = \sum P(X = i, Y = k - i) = \sum P_{X,Y}(i, k-i)$

$$P_S(k) = P(S=k) = \sum_j P(X=j, Y=k-j) = \sum_j P_{X,Y}(j, k-j)$$

If X and Y are independent, then

$$\boxed{P_S(k) = P(S=k) = \sum_j P_X(j) P_Y(k-j) =: P_X * P_Y(k) \quad \text{if } X \text{ \& } Y \text{ are independent \& } S = X+Y}$$

② Suppose that X and Y are jointly continuous. Let $S = X+Y$.

$$f_S(c) = \int_{-\infty}^{+\infty} f_{X,Y}(u, c-u)\,du = \int_{-\infty}^{+\infty} f_{X,Y}(c-v, v)\,dv$$

If X and Y are independent.

$$\boxed{f_S(c) = \int_{-\infty}^{+\infty} f_X(u) f_Y(c-u)\,du = \int_{-\infty}^{+\infty} f_X(c-v) f_Y(v)\,dv = f_X * f_Y(c) \quad \text{if } X \text{ \& } Y \text{ are independent \& } S = X+Y}$$

Week 13:

joint pdf of functions of random variables.

Suppose that X & Y are jointly continuous random variables with joint pdf $f_{X,Y}$.

Suppose that $W = g_1(X,Y)$ and $Z = g_2(X,Y)$. Define $g: \mathbb{R}^2 \to \mathbb{R}^2$ as follows.

We can write.
$$g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}$$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$$

Suppose that (X,Y) is in u–v plane & (W,Z) is in α–β plane.

Let $J(u,v)$ denote the Jacobian matrix of g at point (u,v).

$$J(u,v) = \begin{bmatrix} \frac{\delta g_1(u,v)}{\delta u} & \frac{\delta g_1(u,v)}{\delta v} \\ \frac{\delta g_2(u,v)}{\delta u} & \frac{\delta g_2(u,v)}{\delta v} \end{bmatrix}$$

Proposition: Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$, where $\begin{pmatrix} X \\ Y \end{pmatrix}$ has pdf $f_{X,Y}$, and g is one to one mapping from support of $f_{X,Y}$ to $\mathbb{R}^2$. Suppose that the Jacobian matrix J of g exists, is continuous, and has nonzero determinant everywhere. Then for all (α,β) in the support of $f_{W,Z}$ we have

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(J)|} f_{X,Y}\left(g^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

Important remark: The jacobian matrix J depends on point, i.e., $J = J(u,v)$; however $f_{W,Z}(\alpha,\beta)$ is in terms of α & β. You should calculate J in terms of u and v, and then write it in terms of α and β.

Example:

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad A \text{ is invertible}$$

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

Correlation, Covariance, Correlation Coefficient:

- Let X and Y be random variables with finite second moments:

  the correlation: $E[XY]$

  the covariance: $Cov(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]$

  the correlation coefficient: $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

- Scaling and linearity of Covariance:

  (i) $Cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$

  (ii) $Cov(aX+c, cY+d) = ac\,Cov(X,Y)$

- Important def. & relations.

  ① uncorrelated, negatively correlated, positively correlated

  ② $Var(X) = Cov(X,X)$. In particular if X and Y are uncorrelated

  $$Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(Y,Y) = Var(X) + Var(Y)$$

  ③ If X & Y are independent, they are uncorrelated. The revers does not hold in general.

  (Note: If X & Y are jointly quassian & uncorrelated, they are independent.)

(3) If X & Y are independent, they are uncorrelated. The revers does not hold in general.

(Note: If X & Y are jointly guassian & uncorrelated, they are independent.)

Week 14 & 15:

Estimation & Minimum Mean Square Error (MMSE):

① Sample mean & variance of data set

Given n independent & identically distributed random variables $X_1, X_2, X_3, \dots, X_n$, the sample mean & variance are unbiased estimator and are given as follows:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

② Minimum mean square error estimators of random variable Y, after observing random variable X

. Constant estimator: $\delta^* = E[Y]$, $MMSE = \sigma_Y^2$

. Linear estimator: $L^*(X) = \hat{E}[Y|X] = \mu_Y + \sigma_Y \rho_{X,Y}\left(\frac{X-\mu_X}{\sigma_X}\right)$, $MMSE = \sigma_Y^2(1-\rho_{X,Y}^2)$

. Unconstrained estimator: $g^*(X) = E[Y|X]$, $MMSE = \sigma_Y^2 - E\left[(E[Y|X])^2\right]$

Limit Theorems:

③ Law of large numbers

Suppose that $X_1, X_2, \dots, X_n$ are uncorrelated, each one has the same mean μ, and their variance is bounded by C. Let $S_n = X_1 + X_2 + \dots + X_n$. Then, for any $\delta > 0$, we have

$$P\left(\left\{\left|\frac{S_n}{n} - \mu\right| \geq \delta\right\}\right) = P\left(\left\{w \in \Omega : \left|\frac{S_n(w)}{n} - \mu\right| \geq \delta\right\}\right) \leq \frac{C}{n\delta^2} \xrightarrow{\text{as } n \to \infty} 0$$

④ Central limit theorem

Suppose that $X_1, X_2, \dots, X_n$ are independent, identically distributed, each with mean μ and variance σ^2. Let $S_n = X_1 + X_2 + \dots + X_n$. Then, for any c, we have

$$\lim_{n \to \infty} P\left(\left\{\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq c\right\}\right) = \lim_{n \to \infty} P\left(\left\{\tilde{S}_n \leq c\right\}\right) = \Phi(c)$$

where $\tilde{S}_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{S_n - E[S_n]}{\sqrt{Var(S_n)}}$ is the standardized version of S_n.

Jointly Guassian random variables

We say random variables X and Y are jointly guassian random variables if any linear combination of X and Y are guassian random variables, i.e., for any $a, b \in \mathbb{R}$, $aX + bY$ is a guassian random variables.

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① joint distribution of guassian random variables. Suppose that X, Y are non-degenerate, i.e., they are not linearly related. Then their joint distribution is given by:

$$f_{X,Y}(u,v) = \frac{1}{2\pi\, \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left(\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2(1-\rho^2)}\right)$$

where

$$\mu_X = E[X], \quad \sigma_X^2 = Var(X) \qquad \rho = \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$
$$\mu_Y = E[Y], \quad \sigma_Y^2 = Var(Y)$$

This is calle bivariate normal distribution.

② class of bivariate normal pdfs are preserved via linear transformation:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \text{ has bivariate distribution} \Rightarrow A\begin{pmatrix} X \\ Y \end{pmatrix} \text{ has bivariate distribution}$$

This same as in definition of jointly Guassian random variables:

$$\boxed{X \,\&\, Y \text{ are jointly Guassion} \Rightarrow aX+bY \text{ and } cX+dY \text{ are jointly Guassian}}$$

③ If $\rho = 0$ then $f_{X,Y}(u,v) = f_X(u) f_Y(v)$. Hence uncorrelated jointly guassian random variables are independent

④ If X and Y are independent, then $\rho_{X,Y} = 0$ and their joint distribution $f_{X,Y}(u,v) = f_X(u) f_Y(v)$ is bivariate normal distribution. Hence, independent guassian random variables are jointly guassion.

⑤ If X and Y are guassion, it does not mean that they are jointly guassian.

⑥ If X and Y are jointly guassian, then X has $N(\mu_X, \sigma_X^2)$ distribution and Y has $N(\mu_Y, \sigma_Y^2)$ distribution.

⑦ If X and Y are jointly Guassian, for estimation of Y from X, $L^*(X) = g^*(X)$. Equivalently, $E[Y|X] = \hat{E}[Y|X]$, i.e., the best linear estimator of Y using X, is the best unconstrained estimator.

$$g^*(X) = E[Y|X] = \hat{E}[Y|X] = \mu_Y + \sigma_Y \cdot \rho_{X,Y} \cdot \left(\frac{X-\mu_X}{\sigma_X}\right)$$

⑧ If X and Y are jointly Guassian, the conditional distribution of Y given $X=u$ is $N(\hat{E}[Y|X=u], \sigma_e^2)$

$g(X) = E[Y|X] = \hat{E}[Y|X] = \mu_Y + \sigma_Y \rho_{X,Y} \left(\frac{\quad}{\sigma_X} \right)$

⑧ If X and Y are jointly Guassian, the conditional distribution of Y given $X=u$ is $N(\hat{E}[Y|X=u], \sigma_e^2)$ where σ_e^2 is MSE for $\hat{E}[Y|X]$.

$$\sigma_e^2 = \sigma_Y^2 (1 - \rho_{X,Y}^2)$$

10. **[30 points]** (3 points per answer) → Spring 2015
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose X and Y are independent, continuous-type random variables. $W = \max(X, Y)$, $Z = \min(X, Y)$.

TRUE	FALSE	
☐	☐	$F_W(t) = F_X(t)F_Y(t)$
☐	☐	$F_Z(t) = (1 - F_X(t))(1 - F_Y(t))$
☐	☐	$f_Z(t) = f_X(t)(1 - F_Y(t)) + f_Y(t)(1 - F_X(t))$

(b) Let A, B, C be independent events with $0 < P(A), P(B), P(C) < 1$.

TRUE	FALSE	
☐	☐	$P(AC|B) = P(AC|B^c)$
☐	☐	$P(AB|C) \leq P(C|AB)P(AB)$

(c) Let X, Y be two independent normal random variables with $\mu_X = \mu_Y$ and with $\sigma_X > \sigma_Y$.

TRUE	FALSE	
☐	☐	$\mathbb{P}(Y \leq X) > 1/2$.
☐	☐	$\mathbb{P}(X = Y) > 0$

(d) X and Y are jointly distributed discrete random variables. They are uncorrelated if:

TRUE	FALSE	
☐	☐	$\mathrm{Var}(X + Y) = \mathrm{Var}(X - Y)$
☐	☐	$\mathrm{E}[XY] = 0$
☐	☐	$P(X = u, Y = v) = P(X = u)P(Y = v)$ for every pair (u, v)

7. **[7+16+7 points]** Suppose $X \sim N(1,1)$ and $Y \sim N(1,4)$ are independent Gaussian random variables. Define the random variables $Z = 2X + Y$ and $W = X - Y$. → spring 2019

(a) Find the unconstrained MMSE estimator of Y given X, and the resulting MSE.

(b) Find the unconstrained MMSE estimator of Z given W, and the resulting MSE.

(c) If instead $W = X - aY$ for some real a and $E[Z|W] = E[Z]$, find a.

10. **[10+10 points]** Suppose X and Y are jointly Gaussian with the following parameters: $\mu_x = 0,\ \mu_y = 0,\ \sigma_x^2 = 1,\ \sigma_y^2 = 2^2,\ \rho = 1/8$.

(a) Find $P\{2X + Y \geq 3\}$. Express your answer using the Q function.

(b) Find $E[Y^2|X = 2]$

7. **[15 points]** The two parts of the problem are unrelated. → spring 2018

(a) Suppose a fair die is rolled 100 times. What is a rough approximation to the sum of the numbers showing, based on the law of large numbers?

(b) Suppose each of 1200 real numbers are rounded to the nearest integer and then added. Assume the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. The random variable equal to the sum is denoted by S. Using the CLT, find the approximate probability that the absolute value of the sum of the errors is greater than 5.

3. **[8+4 points]** Suppose X and Y are independent random variables with joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} 2e^{-u}e^{-2v} & \text{if } u \geq 0, v \geq 0 \\ 0 & \text{else.} \end{cases}$$

→ spring 2019

(a) Find the joint pdf of $S = X + Y$ and $W = Y - X$.

(b) Are S and W independent? Explain.