Lecture 4 - 8/29

Monday, August 29, 2022 9:44 AM

Review.

- Set theory. set operations, demorgan's law

  Venn diagram, K-map

- Axioms of probability
  1. Event axioms
  2. Probability axioms

- Counting size of sets:
  1. Principle of counting
  2. Principle of multiplication
  3. Principle of over counting

  $\binom{n}{k}$: # ways to select k members from a set of n different objects

- Probability experiments with equally likely outcomes

  If each member of Ω is equally likely:

  $$P(A) = \frac{|A|}{|\Omega|}, \text{ for any } A \subset \Omega$$

- Frequency interpretation of probability.

  Suppose $\Omega = \{u_1, u_2, \dots, u_m\}$ and we run experiment N times where N is a large number. Suppose:

  N_1 = # of times u_1 was outcome

N_2 = # of times u_2 was outcome

⋮

N_m = # of times u_n was outcome

Notice that $N_1 + N_2 + \cdots + N_m = N$.

frequency of observing u_1 as outcome

$$= \frac{N_1}{N} \approx P(\{u_1\})$$

where "$\approx$" means almost equal to.

Today: ① Random variable, mass function

② Expected value, LOTUS

①Random variable

In rolling a die, the outcome is one of six faces with specific number of dots on it. We assigned numbers to each face for simplicity. In particular, we assigned number to each possible member of sample space.

Def. A random variable is a real-valued function on Ω.

. $X: \Omega \longrightarrow \mathbb{R}$, i.e., a function that assigns real values to members of Ω

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. X is called discrete random variable if

$$\mathbb{P}(X \in \{u_1, u_2, \dots, u_n\}) = 1, \text{ for a finite set } \{u_1, \dots, u_n\}$$

or $\mathbb{P}(X \in \{u_1, u_2, \dots\}) = 1$, for a countably infinite set $\{u_1, u_2, \dots\}$

. Notation convention:

$$P(X \in A) := P(\{\omega \in \Omega \text{ s.t. } X(\omega) \in A\})$$

Def: probability space $(\Omega, \mathcal{F}, P)$ will automatically assign probabilities to random variables:

- If X is a discrete_RV, its probability mass function, p_X, is defined by:

$$p_X(u) = P(X=u) = P(\{\omega \in \Omega \text{ s.t. } X(\omega) = u\})$$

- Notice that by previous def.: $\sum_i p_X(u_i) = 1$

In particular, $p_X(\cdot)$ satisfies axioms of probability.

- Notation convention:

$$p_X(A) = P(X \in A)$$

- Support of pmf $p_X(\cdot)$ is the set of u s.t. $p_X(u) > 0$.

e.1: Consider the experiment of rolling 2 dies:

X_1 = outcome of die one

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$X = X_1 + X_2$

X_2 = outcome of die two

$\Omega = \{(1,1),(1,2),\dots,(1,6),(2,1),\dots,(2,6),\dots,(6,1),(6,2),\dots,(6,6)\}$

Notice that $X_1 : \Omega \longrightarrow \{1,2,3,4,5,6\}$. It's PMF is given by

$$P_{X_1}(1) = P_{X_1}(2) = P_{X_1}(3) = \dots = P_{X_1}(6) = \frac{1}{6}$$

X_2 is simillar to X_1.

For X, we have $X : \Omega \longrightarrow \{2,3,4,\dots,12\}$

$$P_X(2) = P(\{(1,1)\}) = \frac{1}{36}$$

$$P_X(3) = P(\{(1,2),(2,1)\}) = \frac{2}{36}$$

$\vdots$

$$P_X(7) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36}$$

$\vdots$

$$P_X(12) = P(\{(6,6)\}) = \frac{1}{36}$$

e.2: There are ten \$1 and three \$50 bills in a box. We pick five bills randomly. Let

X = total \$ amount

Y = # of \$50 bills.

$X : \Omega \longrightarrow \{5, 54, 103, 152\}$

$Y: \Omega \to \{0, 1, 2, 3\}$

To define Ω, we assign numbers to bills, i.e.,

$$\$1_1, \$1_2, \$1_3, \$1_4, \dots, \$1_{10}$$

$$\$50_1, \$50_2, \$50_3$$

Now, Ω is all possible combinations of bills:

$$\Omega = \{ \{\$1_1, \$1_2, \$1_3, \$1_4, \$1_5\}, \{\$1_1, \$1_2, \$1_3, \$1_4, \$1_6\}, \dots, \{\$50_1, \$50_2, \$50_3, \$1_{12}, \$1_{13}\} \}$$

$$|\Omega| = \binom{13}{5}$$

$$P_X(5) = \frac{\binom{10}{5}}{\binom{13}{5}} = P_Y(0)$$

$$P_X(54) = \frac{\binom{3}{1}\binom{10}{4}}{\binom{13}{5}} = P_Y(1)$$

$\vdots$

$$P_X(152) = \frac{\binom{3}{3}\binom{10}{2}}{\binom{13}{5}} = P_Y(3)$$

Function of random variable:

Suppose that $Y = g(X)$ for some random variable X:

$$P_Y(y) = P(Y=y) = P(\{\omega \in \Omega \text{ s.t. } Y(\omega) = y\})$$

$$= P(\{\omega \in \Omega \text{ s.t. } g(X(\omega)) = y\})$$

$$= \sum_{x : g(x) = y} P(\{\omega \in \Omega \text{ s.t. } X(\omega) = x\})$$

$$= \sum_{x : g(x) = y} P_X(x)$$

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e.3: Toss a fair coin twice. Let $X = \#$ of heads $- \#$ of tails. Let $Y = X^2$.

$\Omega = \{HH, HT, TH, TT\}$.

$X : \Omega \longrightarrow \{-2, 0, 2\}$.

$P_X(-2) = P(X=-2) = P(\{TT\}) = \frac{1}{4}$

$P_X(2) = \frac{1}{4}$, $P_X(0) = \frac{1}{2}$

$Y : \Omega \longrightarrow \{0, 4\}$. $P_Y(0) = P_Y(4) = \frac{1}{2}$

Another way to calculate $P_Y(4)$ is:

$$P_Y(4) = P(Y=4) = \sum_{x : x^2 = 4} P_X(x) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

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② Expected value

Suppose we ran an experiment N times. Consider a random variable $X : \Omega \to \{u_1, u_2, \dots, u_n\}$. Let

$$N_1 = \# \text{ of times } X(\text{outcome}) = u_1$$

$$N_2 = \# \text{ of times } X(\text{outcome}) = u_2$$

$$\vdots$$

$$N_m = \# \text{ of times } X(\text{outcome}) = u_n$$

Now, the average of numbers we have seen

$$= \frac{u_1 N_1 + u_2 N_2 + \cdots + u_n N_n}{N}$$

$$= \sum_i u_i \frac{N_i}{N} \approx \sum_i u_i P_X(u_i)$$

Def. Mean of random variable X (also called expected value) with pmf $P_X(\cdot)$ is denoted by $E[X]$ and is defined by

$$E[X] = \sum_i u_i P_X(u_i).$$

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e.4. In the previous example:

$$E[X] = 0 \cdot P_X(0) + 2 \cdot P_X(2) - 2 P_X(-2) = 0$$

$$E[Y] = 0 \cdot P_Y(0) + 4 \cdot P_Y(4) = 2$$

LOTUS: law of the unconscious statistician:

If there is a random variable Y defined as $Y = g(X)$, for some random variable X, then, if X is discrete, we have

$$E[Y] = E[g(X)] = \sum_i g(u_i) P_X(u_i)$$

To see why:

$$E[Y] = \sum_y y P_Y(y)$$

$$= \sum_y y \sum_{x : g(x) = y} P_X(x)$$

$$= \sum_{x,y : g(x) = y} g(x) \cdot P_X(x) = \sum_x g(x) P_X(x)$$

e.5: Using LOTUS in e.4, we have

$$E[Y] = E[X^2] = 0^2 \cdot P_X(0) + (-2)^2 P_X(-2) + 2^2 P_X(2) = 4.$$