Lecture 35 - 11/16

Wednesday, November 16, 2022 9:53 AM

Review:

Suppose that X & Y are jointly continuous random variables with joint pdf $f_{X,Y}$.

Suppose that $W = g_1(X,Y)$ and $Z = g_2(X,Y)$. Define $g: \mathbb{R}^2 \to \mathbb{R}^2$ as follows.

We can write. $$g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}$$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$$

Suppose that (X,Y) is in u–v plane & (W,Z) is in α–β plane.

Let $J(u,v)$ denote the Jacobian matrix of g at point (u,v).

$$J(u,v) = \begin{bmatrix} \frac{\delta g_1(u,v)}{\delta u} & \frac{\delta g_1(u,v)}{\delta v} \\ \frac{\delta g_2(u,v)}{\delta u} & \frac{\delta g_2(u,v)}{\delta v} \end{bmatrix}$$

Proposition: Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$, where $\begin{pmatrix} X \\ Y \end{pmatrix}$ has pdf $f_{X,Y}$, and g is one to one mapping from support of $f_{X,Y}$ to $\mathbb{R}^2$. Suppose that the Jacobian matrix J of g exists, is continuous, and has nonzero determinant everywhere. Then for all (α,β) in the support of $f_{W,Z}$ we have

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(J)|} f_{X,Y}\left(g^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

Important remark: The jacobian matrix J depends on point, i.e. $J = J(u,v)$; however $f_{W,Z}(\alpha,\beta)$ is in terms of α & β. You should calculate J in terms of u and v, and then write it in terms of α and β.

Example:

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad A \text{ is invertible}$$

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{XY}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

Today:

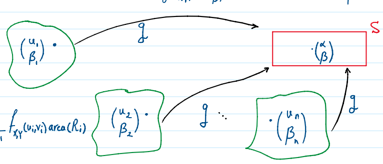
- Transformation of pdfs under a many-to-one mapping
- Correlation & covariance

- Transformation of pdfs under a many-to-one mapping

Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$, and $g: \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that g has a continuous Jacobian with nonzero determinant. Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix}$ lives in α–β plane and $\begin{pmatrix} X \\ Y \end{pmatrix}$ lives in u–v plane. Fix $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and consider a small rectangle around $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Let S denote this rectangle. Notice that

$$P\left(\begin{pmatrix} W \\ Z \end{pmatrix} \in S\right) \approx f_{W,Z}(\alpha,\beta) \cdot \text{area}(S) \qquad \boxed{\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}\ S$$

Let $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \dots, \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ denote the set of all values for which $g\left(\begin{pmatrix} u_i \\ v_i \end{pmatrix}\right) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Let R_i denote the shape around $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ that is mapped to S. We have:



$$P((\alpha,\beta) \in S)$$
$$= P\left((u,v) \in \bigcup_{i=1}^{n} R_i\right)$$
$$= \sum_{i=1}^{n} P((u,v) \in R_i) \approx \sum_{i=1}^{n} f_{XY}(u_i,v_i)\,\text{area}(R_i)$$

Notice that if S is small enough then R_is are disjoint. Moreover, $\frac{\text{area}(R_i)}{\text{area}(S)} \approx \frac{1}{|\det(J(u_i,v_i))|}$

Hence, we have

$$f_{W,Z}(\alpha,\beta) = \sum_{(u_i,v_i):\, g\left(\begin{pmatrix} u_i \\ v_i \end{pmatrix}\right) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}} \frac{f_{XY}(u_i,v_i)}{|\det(J(u_i,v_i))|}$$

Notice that after calculating r.h.s. you need to write it in terms of α and β.

Example 4.7.8 Suppose $W = \min\{X, Y\}$ and $Z = \max\{X, Y\}$, where X and Y are jointly continuous-type random variables. Express $f_{W,Z}$ in terms of $f_{X,Y}$.

Solution.

Approach 1. Notice that $P(W=Z) = P(X=Y) = 0$. Hence, we focus on the set $\{w \in \Omega : W(\omega) \neq Z(\omega)\}$.

Suppose that $\alpha < \beta$. Notice that $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} \min(X,Y) \\ \max(X,Y) \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$. Hence, if $g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ then either $\alpha = u, \beta = v$ or $\alpha = v, \beta = u$.

That is $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ and $g\left(\begin{pmatrix} u_i \\ v_i \end{pmatrix}\right) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Next, we need to calculate the Jacobian matrix at $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$.

Notice that around $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} u \\ v \end{pmatrix}$ and near $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$, $g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} v \\ u \end{pmatrix}$. Hence,

$$J(u_1,v_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J(u_2,v_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in either case $|\det(J(u_i,v_i))| = 1$. Hence, $f_{W,Z}(\alpha,\beta) = f_{XY}(u_1,v_1) + f_{XY}(u_2,v_2) = f_{XY}(\alpha,\beta) + f_{XY}(\beta,\alpha)$

Approach 2. See the solution in the book.

- Correlation and covariance

Let X and Y be random variables with finite second moments.

the correlation. $E[XY]$

the covariance. $Cov(X,Y) = E\big[(X-E[X])(Y-E[Y])\big]$

the correlation coefficient. $\rho_{XY} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

Important remarks.

① Notice that $Cov(X,X) = Var(X)$, i.e, covariance generalizes variance to pair of random variables.

② Simillar to variance, we have $Cov(X,Y) = E[XY] - E[X]E[Y]$.

Def. We say X and Y are uncorrelated if $Cov(X,Y) = 0$. Notice that if $Var(X)>0$ and $Var(Y)>0$, then being uncorrelated is equivalent to having $\rho_{X,Y} = 0$.

Def. We say X and Y are positively correlated if $Cov(X,Y)>0$ ($\rho_{X,Y}>0$).

Def. We say X and Y are negatively correlated if $Cov(X,Y)<0$ ($\rho_{X,Y}<0$).

Important remark.

X and Y are independent $\Longrightarrow$ $\rho_{XY} = 0$, $Cov(X,Y)=0$, uncorrelated

$\rho_{XY} = 0$, $Cov(X,Y)=0$, uncorrelated does not imply that X and Y are independent (unless they are jointly guassian)

Def. We say $X_1, X_2, \dots, X_n$ are uncorrelated if they are pairwise uncorrelated.

Properties.

① linearity of covariance.

$cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$

$cov(aX+b, cY+d) = ac\,Cov(X,Y)$

② If X and Y are uncorrelated.

$Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(Y,Y) = Var(X) + Var(Y)$

4.17. **[Deducing a covariance from variances]**
Consider random variables X and Y on the same probability space.

(a) If $\mathrm{Var}(X+2Y) = 40$ and $\mathrm{Var}(X-2Y) = 20$, what is $\mathrm{Cov}(X,Y)$?

(b) In part (a), determine $\rho_{X,Y}$ if $\mathrm{Var}(X) = 2 \cdot \mathrm{Var}(Y)$.

Solution.

(a) Notice that

$Var(X+2Y) = cov(X+2Y, X+2Y) = cov(X,X) + 2cov(X,Y) + 2cov(Y,X) + 4Cov(Y,Y)$

$= Var(X) + 4Var(Y) + 4Cov(X,Y)$

$Var(X-2Y) = Var(X) + 4Var(Y) - 4Cov(X,Y)$

$\Rightarrow Var(X+2Y) - Var(X-2Y) = 8Cov(X,Y) \Rightarrow Cov(X,Y) = \frac{20}{8} = 2.5$

(b) Notice that $20 = Var(X-2Y) = Var(X) + 4Var(Y) - 4cov(X,Y) = 6Var(Y) - 10$

$\Rightarrow Var(Y) = 5$ and $Var(X) = 10$.

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{2.5}{\sqrt{50}}$$