Lecture 34 - 11/14
Friday, November 11, 2022

Today:
1. Joint pdf of functions of random variables.
2. Transformation of pdfs under linear mapping.
3. Transformation of pdfs under one-to-one mapping.

Suppose that $X$ and $Y$ are jointly continuous with pdf $f_{XY}$. Suppose that $W = aX + bY$ and $Z = cX + dY$ for some constants $a, b, c, d$. Our goal is to find $f_{W,Z}$. We can write:

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Note that if $\mathbf{A}$ is invertible, then $W$ & $Z$ are linearly related to each other, i.e.,

- $\mathbf{A}$ invertible $\Rightarrow$ e.g. $W = \frac{a}{c}Z$
- $\mathbf{A}$ invertible $\Rightarrow$ e.g. $W = \frac{b}{d}Z$

All in all, given $\mathbf{A}$ is invertible, we have:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} W \\ Z \end{pmatrix}$$

Suppose $(X,Y)$ is in $u-v$ plane, and $(W,Z)$ is in $a-b$ plane. We have:

$$P(W \leq u, Z \leq v) = P\left(\begin{pmatrix} X \\ Y \end{pmatrix} \leq \begin{pmatrix} u \\ v \end{pmatrix}\right) = \iint_{\mathbf{A}^{-1}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)} f_{XY}(u,v) \, du \, dv$$

- change of variable $(X,Y) = A^{-1}(u,v)$, we have $du \, dv = |\text{det}(\mathbf{A})| \, dx \, dy$

Proposition: Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = \mathbf{A} \begin{pmatrix} X \\ Y \end{pmatrix}$, where $(X,Y)$ has pdf $f_{XY}$, and $\mathbf{A}$ is a matrix with $\text{det}(\mathbf{A}) \neq 0$. Then $(W,Z)$ has joint pdf given by:

$$f_{W,Z}(w,z) = \frac{1}{|\text{det}(\mathbf{A})|} f_{XY}\left(\mathbf{A}^{-1}\begin{pmatrix} w \\ z \end{pmatrix}\right)$$

Important remarks:

- $\mathbf{A}$ is invertible if and only if $\text{det}(\mathbf{A}) \neq 0$. In particular:
  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a^* & 1 \\ \frac{d-b}{d} & -b \end{pmatrix}$$
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A' = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

The change of variable formula \( \int_{\mathcal{B}} f_X(u) \, du \) can be interpreted in terms of area.

Area of a rectangle \( R \) in \( u-v \) plane = height of \( R \) \times \text{length of } R

Now let's pass \( R \) via the linear map \((p) \mapsto A(p)^T \), i.e., we replace each point \((p)\) with \( A(p)^T \).

Then \( \text{area}(S) = |\det A| \cdot \text{area}(R) \)

Another interpretation of proposition is as follows. Let \( R \) denote a small rectangle in \( u-v \) plane, and let \( S \) denote its image via \( (p) \mapsto A(p)^T \). We have

\[ \int_{\mathcal{U}} f_X(x) \cdot \det(A) \, dx = \int_{\mathcal{V}} f_Y(y) \, dy \]

where \((x) \in R \) and \((y) \in S \).

**Example 4.7.2** Suppose \( X \) and \( Y \) have joint pdf \( f_{X,Y} \), and \( W = X - Y \) and \( Z = X + Y \). Express the joint pdf of \( W \) and \( Z \) in terms of \( f_{X,Y} \).

---

Transformation of pdfs under a one to one mapping.

Now, suppose that \( W = g(X,Y) \) and \( Z = h(X,Y) \). We can write

\[ (W,Z) = g((X,Y)) \]

where \( g \) maps \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is given by \( g((x,y)) = (g_{100},g_{010}) \). Suppose that \( g \) is invertible.

Following the same line of reasoning,

\[ P(W \leq w, Z \leq z) = P(g((X,Y)) \leq (w,z)) \]

\[ = \int_{g((X,Y)) \leq (w,z)} f_{X,Y}(x,y) \, dx \, dy \]

\[ \text{change of variable } (p) = g((x,y)) \quad (\text{why}) \]

we have \( dp = |\det J| \, dp \)

where \( J \) is the Jacobian matrix.
we have \( dV = |\det(J)| d(x,y) \),

where \( J \) is the Jacobian matrix.

Proposition. Suppose that \((X,Z) = g((Y,\lambda))\), where \((Y,\lambda)\) has pdf \(f_{Y,\lambda}\), and \(g\) is one to one mapping from support of \(f_{Y,\lambda}\) to \(\mathbb{R}^2\). Suppose that the Jacobian matrix \(J\) of \(g\) exists, is continuous, and has nonzero determinant everywhere. Then for all \((w,\lambda)\) in the support of \(f_{W,Z}\), we have

\[
f_{W,Z}(w,\lambda) = \frac{1}{|\det(J)|} f_{X,Y}(A^T((w,\lambda)))
\]

Important remarks:

1. The Jacobian matrix at point \((w,\lambda)\) is defined as

\[
J = J_{w,\lambda} = \begin{pmatrix}
\frac{\partial Y}{\partial w} & \frac{\partial Y}{\partial \lambda} \\
\frac{\partial \lambda}{\partial w} & \frac{\partial \lambda}{\partial \lambda}
\end{pmatrix}
\]

The Jacobian matrix is also called derivative of \(g\); we can approximate values of \(g\) around \((w,\lambda)\) by

\[
g\left((w,\lambda)\right) \approx g\left((w,\lambda)\right) + J_{w,\lambda}(u,v) \left((w,\lambda) - (w,\lambda)\right)
\]

This is similar to functions of one variable, i.e., given \(f: \mathbb{R} \rightarrow \mathbb{R}\), we can approximate values of \(f\) around \(x\) by

\[
f(u) \approx f(x) + f'(x)(u-x)
\]

2. Notice that in the Jacobian matrix, first column is derivative with respect to \(w\). The reason is clear from \((\ast)\).

3. First column of \(J_{w,\lambda}\) is multiplied by \(w-u, \lambda-v\) in \((\ast)\).

4. The Jacobian matrix \(J\) depends on point, i.e., \(J = J_{w,\lambda}\); however in \((\ast\ast)\) integration is with respect to \(\alpha, \beta\). You should calculate \(J\) in terms of \(u, v\), and then write it in terms of \(\alpha, \beta\).

5. The same one interpretation also applies here.

\[
f_{Z}(w,\alpha) = P((\alpha,\beta) e R) = P((W,Z) e S) = f_{W,Z}(w,\alpha,\beta)
\]

where \(R\) is a small rectangle at point \((w,\alpha)\) and \(S\) is its image via \(g\), i.e., \(g(R) = S\).

We have

\[
\text{area}\,(S) = |\det(J)| \cdot \text{area}\,(R) = |\det(J_{w,\lambda})| \cdot \text{area}\,(R)
\]

6. If function \(g\) is linear, i.e., \((X,Z) = A(Y)\) then Jacobian of \(g\) is the fixed matrix \(A\).

**Example 4.7.5** Let \(X, Y\) have the joint pdf:

\[
f_{X,Y}(u, v) = \begin{cases}
    u + v & (u,v) \in [0,1]^2 \\
    0 & \text{else}
\end{cases}
\]

and let \(W = X^2\) and \(Z = X(1 + Y)\). Find the pdf, \(f_{W,Z}\).