Lecture 34 - 11/14

Friday, November 11, 2022 12:33 PM

Today: joint pdf of functions of random variables

① Transformation of pdfs under linear mapping.

② Transformation of pdfs under a one to one mapping.

① Transformation of pdfs under linear mapping.

Suppose that X and Y are jointly-continuous with pdf $f_{X,Y}$. Suppose that $W = aX + bY$ and $Z = cX + dY$ for some constants a, b, c, d. Our goal is to find $f_{W,Z}$. We can write:

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Note that if A is not invertible, then W & Z are linearly related to each other, i.e.,

$$A \text{ not invertible } \& \ c \neq 0 \Rightarrow W = \frac{a}{c} Z$$

$$A \text{ not invertible } \& \ d \neq 0 \Rightarrow W = \frac{b}{d} Z$$

All in all, given A is invertible, we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A^{-1} \begin{pmatrix} W \\ Z \end{pmatrix}$$

Suppose (X,Y) is in $u-v$ plane, and (W,Z) is in $\alpha-\beta$ plane. We have,

$$P(W \leq e, Z \leq f) = P\left(A\begin{pmatrix} X \\ Y \end{pmatrix} \leq \begin{pmatrix} e \\ f \end{pmatrix}\right)$$

$$= \iint_{A\binom{u}{v} \leq \binom{e}{f}} f_{X,Y}(u,v)\, dv\, du$$

$$= \int_{-\infty}^{e}\int_{-\infty}^{f} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \frac{d\beta\, d\alpha}{|\det(A)|}$$

change of variable $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$

we have $d\alpha\, d\beta = |\det(A)|\, du\, dv$

Proposition: Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$, where $\begin{pmatrix} X \\ Y \end{pmatrix}$ has pdf $f_{X,Y}$, and A is a matrix with $\det(A) \neq 0$. Then $\begin{pmatrix} W \\ Z \end{pmatrix}$ has joint pdf given by

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

Important remarks:

① A is invertible if & only if $\det(A) \neq 0$. In particular,

$$A = \begin{pmatrix} a & b \\ \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{} \begin{pmatrix} d & -b \\ \end{pmatrix}$$

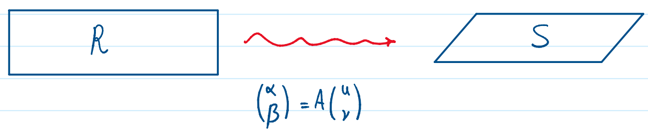
③ A is invertible if and only if $\det(A) \neq 0$. In particular,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

② The change of variable formula $d\alpha\, d\beta = |\det(A)|\, du\, dv$ can be interpreted in terms of area:

Area of a rectangle R in $u-v$ plane = height of R x length of R

Now lets pass R via the linear map $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$, i.e, we replace each point $\begin{pmatrix} u \\ v \end{pmatrix}$ with $A\begin{pmatrix} u \\ v \end{pmatrix}$:



Then $\text{area}(S) = |\det A|\, \text{area}(R)$

③ Another interpretation of proposition is as follows: let R denote a small rectangles in $u-v$ plane, and let S denote its image via $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$. We have

$$f_{X,Y}(u,v) \cdot \text{area}(R) \approx P((X,Y) \in R) = P((W,Z) \in S) \approx f_{W,Z}(\alpha,\beta) \cdot \text{area}(S)$$

where $(u,v) \in R$ & $(\alpha,\beta) \in S$.

Example 4.7.2 Suppose X and Y have joint pdf $f_{X,Y}$, and $W = X - Y$ and $Z = X + Y$. Express the joint pdf of W and Z in terms of $f_{X,Y}$.

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② Transformation of pdfs under a one to one mapping.

Now, suppose that $W = g_1(X,Y)$ and $Z = g_2(X,Y)$. We can write

$$\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$$

where $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}$. Suppose that g is invertible.

Following the same line of reasoning:

$$P(W \leq e, Z \leq f) = P\left(g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) \leq \begin{pmatrix} e \\ f \end{pmatrix}\right)$$

$$= \iint_{g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \leq \begin{pmatrix} e \\ f \end{pmatrix}} f_{X,Y}(u,v)\, dv\, du$$

$$= \int_{-\infty}^{e}\int_{-\infty}^{f} f_{X,Y}\left(g^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \frac{d\beta\, d\alpha}{|\det J|} \rightsquigarrow \text{change of variable } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = g\begin{pmatrix} u \\ v \end{pmatrix} \quad (**)$$

we have $d\alpha\, d\beta = |\det(J)|\, du\, dv$

where J is the jacobian matrix.

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Proposition: Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$, where $\begin{pmatrix} X \\ Y \end{pmatrix}$ has pdf $f_{X,Y}$, and g is one to one mapping from support of $f_{X,Y}$ to $\mathbb{R}^2$. Suppose that the Jacobian matrix J of g exists, is continuous, and has nonzero determinant everywhere. Then for all (α,β) in the support of $f_{W,Z}$ we have

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

Important remarks:

① The jacobian matrix at point (u,v) is defined as

$$J = J(u,v) = \begin{pmatrix} \frac{\delta g_1(u,v)}{\delta u} & \frac{\delta g_1(u,v)}{\delta v} \\ \frac{\delta g_2(u,v)}{\delta u} & \frac{\delta g_2(u,v)}{\delta v} \end{pmatrix}$$

The jacobian matrix is also called derivative of g; we can approximate values of g around $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ by

$$g\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \approx g\left(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right) + J(u_0,v_0)\left(\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right) \quad (*)$$

This is similar to functions of one variable, i.e., given $f: \mathbb{R} \to \mathbb{R}$ we can approximates values of f around u_0 by

$$f(u) \approx f(u_0) + f'(u_0)(u - u_0)$$

② Notice that in the jacobian matrix first column is derivative with respect to u. The reason is clear from $(*)$. First column of $J(u,v)$ is multiplied by $u-u_0$ in $(*)$.

③ The jacobian matrix J depends on point, i.e., $J = J(u,v)$; however in $(**)$ integration is with respect to α & β. You should calculate J in terms of u and v, and then write it in terms of α and β.

④ The same area interpretation also applies here.

$$f_{X,Y}(u,v) \cdot \text{area}(R) \approx P((X,Y) \in R) = P((W,Z) \in S) \approx f_{W,Z}(\alpha,\beta) \cdot \text{area}(S)$$

where R is a small rectangle at point (u,v) and S is its image via g, i.e., $g(R) = S$.

We have: $\quad \text{area}(S) = |\det(J)|\, \text{area}(R) = |\det(J(u,v))|\, \text{area}(R)$

⑤ If function g is linear, i.e. $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$ then Jacobian of g is the fixed matrix A.

Example 4.7.5 Let X, Y have the joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} u+v & (u,v) \in [0,1]^2 \\ 0 & \text{else} . \end{cases}$$

and let $W = X^2$ and $Z = X(1+Y)$. Find the pdf, $f_{W,Z}$.