Example 4.6.4 Consider the following variation of the Buffon’s needle problem (Example 4.6.3). Suppose a needle of unit length is thrown at random onto a plane with both a vertical grid and a horizontal grid, each with unit spacing. Find the probability the needle, after it comes to rest, does NOT intersect any grid line.

Note: by “thrown at random onto a plane with both a vertical & horizontal grid”, we mean the lower end of the needle is uniformly distributed over \([0,1] \times [0,1]\) & its angle with the vertical line is uniformly distributed over \([0,\pi]\).

\[ V \sim \operatorname{Unif}([0,1]) \]
\[ U \sim \operatorname{Unif}([0,1]) \]

\[
\sin(\theta) \quad U \quad \cos(\theta)
\]

Solution: Not crossing the grid means:

\[ U > \sin(\theta) \quad \text{and} \quad V > \cos(\theta) \quad \text{for} \quad \theta \in \left[0, \frac{\pi}{2}\right] \]

\[ U > \sin(\theta) \quad \text{and} \quad V + |\cos(\theta)| < 1 \quad \text{for} \quad \theta \in \left[\frac{\pi}{2}, \pi\right] \]

Notice that \(U\) & \(V\) are independent of each other.

Let \(M_h\) denote the event of not crossing a horizontal line \& \(M_v\), the event of not crossing a vertical line.

\[
P(M_h \cap M_v) = \int_0^\pi f_\theta(\theta) P(M_h, M_v \mid \theta, x) \, d\theta
\]

\[= \int_0^\pi \frac{1}{\pi} P(M_h, M_v \mid \theta, x) \, d\theta \]

\[= \frac{1}{\pi} \int_0^{\pi/2} P(U > \sin(\theta), V > \cos(\theta)) \, d\theta + \frac{1}{\pi} \int_{\pi/2}^\pi P(U > \sin(\theta), V + |\cos(\theta)| < 1) \, d\theta \]

\[= \frac{1}{\pi} \int_0^{\pi/2} (1 - \sin(\theta)) (1 - \cos(\theta)) \, d\theta + \frac{1}{\pi} \int_{\pi/2}^\pi (1 - \sin(\theta)) (1 - |\cos(\theta)|) \, d\theta \]
\[ \frac{3}{\pi} \left( \frac{1}{x} \right) = L \frac{3}{\pi}. \]

**Example 4.6.5** Observations \( X_1, \ldots, X_T \) produced by a drone’s altimeter are assumed to have the form \( X_t = bt + W_t \) for \( 1 \leq t \leq T \), where \( b \) is an unknown constant representing the rate of ascent of the drone (if \( b < 0 \) it means the drone is descending) and \( W_1, \ldots, W_T \) represent observation noise and are assumed to be independent, \( N(0,1) \) random variables. Obtain the maximum likelihood estimator of \( b \) for a particular vector of observations \( u_1, \ldots, u_T \). An estimator is called **unbiased** if the mean of the estimator is equal to the parameter that is being estimated. Determine if the ML estimator of \( b \) is unbiased.

**Problem:** Suppose that \( X \) & \( Y \) are continuous-type and independent. Show that \( P(X = Y) = 0 \).

**Solution:** Recall that \( P((X,Y) \in A) = \int \int f_{X,Y}(u,v) \, dv \, du \). Here \( A = \{(x,y) \in \mathbb{R}^2 : x = y\} \) is a line & has 1 dimension. Hence \( P((X,Y) \in A) = 0 \).

Another way to see this:

\[
P(X = Y) = \lim_{\varepsilon \to 0} P(|X - Y| < \varepsilon) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \int_{u-\varepsilon}^{u+\varepsilon} f_{X,Y}(u,v) \, dv \, du
\]
\[
= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \left[ \int_{u-\varepsilon}^{u+\varepsilon} f_{X}(u) f_{Y}(v) \, dv \right] \, du
= \int_{-\infty}^{\infty} \left[ \lim_{\varepsilon \to 0} \int_{u-\varepsilon}^{u+\varepsilon} f_{Y}(v) \, dv \right] \, du = 0.
\]

Today: joint pdf of functions of random variables

Transformation of pdfs under a linear mapping

Suppose that \( X \) and \( Y \) are jointly continuous with pdf \( f_{X,Y} \). Suppose that \( W = aX + bY \) and \( Z = cX + dY \) for some constants \( a, b, c, d \). Our goal is to find \( f_{W,Z} \). We can write:

\[
\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}
\]

where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

If \( A \) is invertible, we have

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = A^{-1} \begin{pmatrix} W \\ Z \end{pmatrix}
\]

Suppose \((X,Y)\) is in \( u-v\) plane, and \((W,Z)\) is in \( a-b\) plane. We have.
\[ P(W \leq s, Z \leq t) = P\left( (X, Y) \leq A'(\begin{pmatrix} s \\ t \end{pmatrix}) \right) \]

\[ = \int \int f_{X,Y}(u,v) \, du \, dv \]

\[ (u, v) \in A'(\begin{pmatrix} s \\ t \end{pmatrix}) \]

\[ = \int \int f_{X,Y}(A'(\begin{pmatrix} s \\ t \end{pmatrix})) \, \frac{d\beta}{|\det(A)|} \quad \text{change of variable \( (\begin{pmatrix} s \\ t \end{pmatrix}) = A(\begin{pmatrix} u \\ v \end{pmatrix}) \)} \]

we have \( d\alpha d\beta = |\det(A)| \, du \, dv \)

**Proposition.** Suppose that \(\begin{pmatrix} W \\ Z \end{pmatrix} = A(\begin{pmatrix} X \\ Y \end{pmatrix})\), where \(\begin{pmatrix} X \\ Y \end{pmatrix}\) has pdf \(f_{X,Y}\), and \(A\) is a matrix with \(\det(A) \neq 0\). Then \(\begin{pmatrix} W \\ Z \end{pmatrix}\) has joint pdf given by

\[ f_{W,Z}(w,z) = \frac{1}{|\det(A)|} f_{X,Y}(A^{-1}(\begin{pmatrix} w \\ z \end{pmatrix})) \]

If \(A\) is not invertible, Then \(W\) and \(Z\) are linearly related to each other, i.e., \(W = \frac{a}{c} Z\).

In this case, it only makes sense to talk about pdf of \(W\).