Example 4.6.4 Consider the following variation of the Buffon’s needle problem (Example 4.6.3). Suppose a needle of unit length is thrown at random onto a plane with both a vertical grid and a horizontal grid, each with unit spacing. Find the probability the needle, after it comes to rest, does NOT intersect any grid line.

Note. by “thrown at random onto a plane with both a vertical & horizontal grid”, we mean the lower end of the needle is uniformly distributed over $[0,1] \times [0,1]$ & its angle with the vertical line is uniformly distributed over $[0,\pi]$.

$$V \sim \text{Unif}([0,1])$$
$$U \sim \text{Unif}([0,1])$$

Solution. Not crossing the grid means:

$$U \geq \sin(\theta) \quad \text{and} \quad V \geq \cos(\theta) \quad \text{for} \quad \theta \in \left[0, \frac{\pi}{2}\right]$$

$$U \geq \sin(\theta) \quad \text{and} \quad V + |\cos(\theta)| \leq 1 \quad \text{for} \quad \theta \in \left[\frac{\pi}{2}, \pi\right]$$

Notice that $U$ & $V$ are independent of each other.

Let $M_h$ denote the event of not crossing a horizontal line & $M_v$ the event of not crossing a vertical line.

$$P(M_h^c M_v^c) = \int_{\theta}^{\pi} \frac{1}{\pi} P(M_h^c M_v^c | \theta) \, d\theta$$

$$= \int_{0}^{\pi} \frac{1}{\pi} P(M_h^c M_v^c | \theta) \, d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\pi} P(U \geq \sin(\theta), V \geq \cos(\theta)) \, d\theta + \frac{1}{\pi} \int_{\pi/2}^{\pi} P(U \geq \sin(\theta), V + |\cos(\theta)| \leq 1) \, d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\pi} (1 - \sin(\theta))(1 - \cos(\theta)) \, d\theta + \frac{1}{\pi} \int_{\pi/2}^{\pi} (1 - \sin(\theta))(1 - |\cos(\theta)|) \, d\theta$$
Example 4.6.5 Observations \( X_1, \ldots, X_T \) produced by a drone’s altimeter are assumed to have the form \( X_t = bt + W_t \) for \( 1 \leq t \leq T \), where \( b \) is an unknown constant representing the rate of ascent of the drone (if \( b < 0 \) it means the drone is descending) and \( W_1, \ldots, W_T \) represent observation noise and are assumed to be independent, \( N(0,1) \) random variables. Obtain the maximum likelihood estimator of \( b \) for a particular vector of observations \( u_1, \ldots, u_T \). An estimator is called *unbiased* if the mean of the estimator is equal to the parameter that is being estimated. Determine if the ML estimator of \( b \) is unbiased.

**Problem:** Suppose that \( X \) & \( Y \) are continuous type and independent. Show that \( P(X = Y) = 0 \).

**Solution:** Recall that \( P((X,Y) \in A) = \iint_{A} f_{X,Y}(u,v) \, dv \, du \). Here \( A = \{(x,y) \in \mathbb{R}^2 : x = y\} \) is a line & has 1 dimension. Hence \( P((X,Y) \in A) = 0 \).

Another way to see this,

\[
P(X = Y) = \lim_{\varepsilon \to 0} P(|X-Y| < \varepsilon) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \int_{u-\varepsilon}^{u+\varepsilon} f_{X,Y}(u,v) \, dv \, du
\]

\[
= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \int_{u-\varepsilon}^{u+\varepsilon} f_X(u) f_Y(v) \, dv \, du = \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \int_{u-\varepsilon}^{u+\varepsilon} f_Y(v) \, dv \, du = 0.
\]

---

Today: joint pdf of functions of random variables.

- Transformation of pdfs under a linear mapping.

Suppose that \( X \) and \( Y \) are jointly continuous with pdf \( f_{X,Y} \). Suppose that \( W = aX + bY \) and \( Z = cX + dY \) for some constants \( a,b,c,d \). Our goal is to find \( f_{W,Z} \). We can write,

\[
\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}
\]

where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

If \( A \) is invertible, if \( A \) is invertible, we have

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = A^{-1} \begin{pmatrix} W \\ Z \end{pmatrix}
\]

Suppose \((X,Y)\) is in \( u-v \) plane, and \((W,Z)\) is in \( \alpha-\beta \) plane. We have.
\[ P(W < t, Z < f) = P\left( A(X) < \left( \begin{array}{c} t \\ f \end{array} \right) \right) \]

\[ = \iint_{A(X) < \left( \begin{array}{c} t \\ f \end{array} \right)} f_{X,Y}(u,v) \, du \, dv \]

\[ = \iint_{\infty}^{t} f_{X,Y}(A^{-1}(\beta)) \, d\beta \, du \quad \text{change of variable } (\beta) = A^{-1}(u) \]

we have \[ d\beta = |dct(A)| \, du \, dv \]

**Proposition.** Suppose that \( \left( \begin{array}{c} W \\ Z \end{array} \right) = A \left( \begin{array}{c} X \\ Y \end{array} \right) \), where \( \left( \begin{array}{c} X \\ Y \end{array} \right) \) has pdf \( f_{X,Y} \), and \( A \) is a matrix with \( \text{det}(A) \neq 0 \). Then \( \left( \begin{array}{c} W \\ Z \end{array} \right) \) has joint pdf given by

\[ f_{W,Z}(w,z) = \frac{1}{|\text{det}(A)|} f_{X,Y}(A^{-1}(w)) \]

If \( A \) is not invertible, then \( W \) and \( Z \) are linearly related to each other, i.e., \( W = \frac{a}{c} Z \).

In this case, it only makes sense to talk about pdf of \( W \).