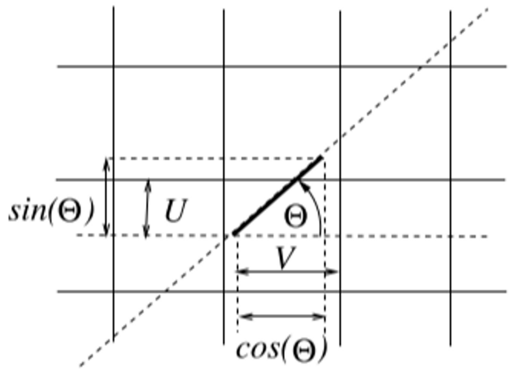
Example 4.6.4 Consider the following variation of the Buffon's needle problem (Example 4.6.3). Suppose a needle of unit length is thrown at random onto a plane with both a vertical grid and a horizontal grid, each with unit spacing. Find the probability the needle, after it comes to rest, does NOT intersect any grid line.

Note: by "thrown at random onto a plane with both a vertical & horizontal grid", we mean the lower end of the needle is uniformly distributed over $[0,1]\times[0,1]$ & its angle with the the vertical line is uniformly distributed over $[0,\pi]$.



$V \sim Unif([0,1])$

$U \sim Unif([0,1])$

Solution: Not crossing the grid means.

$$U \geq Sin(\theta) \text{ and } V \geq Cos(\theta), \text{ for } \theta \in [0, \tfrac{\pi}{2}]$$

$$U > Sin(\theta) \text{ and } V + |cos(\theta)| \leq 1 \text{ for } \theta \in [\tfrac{\pi}{2}, \pi]$$

Notice that U & V are independent of each other.

Let M_h denote the event of not crossing a horizontal line & M_v the event of not crossing a vertical line.

$$P(M_h M_v) = \int_0^\pi f_\Theta(\theta)\, P(M_h M_v | \Theta = \theta)\, d\theta$$

$$= \int_0^\pi \frac{1}{\pi} P(M_h M_v | \Theta = \theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi/2} P(U \geq Sin(\theta), V \geq cos(\theta))\, d\theta + \frac{1}{\pi}\int_{\pi/2}^{\pi} P(U \geq Sin(\theta), V + |Cos(\theta)| \leq 1)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (1 - Sin(\theta))(1 - cos(\theta))\, d\theta + \frac{1}{\pi}\int_{\pi/2}^{\pi} (1 - Sin(\theta))(1 - |cos(\theta)|)\, d\theta$$

$= \frac{2}{\pi}\left(\frac{\pi}{2} - 1 - 1 + \frac{1}{2}\right) = 1 - \frac{3}{\pi}.$

Example 4.6.5 Observations $X_1, \ldots, X_T$ produced by a drone's altimeter are assumed to have the form $X_t = bt + W_t$ for $1 \le t \le T$, where b is an unknown constant representing the rate of ascent of the drone (if $b < 0$ it means the drone is descending) and $W_1, \ldots, W_T$ represent observation noise and are assumed to be independent, $N(0,1)$ random variables. Obtain the maximum likelihood estimator of b for a particular vector of observations $u_1, \ldots, u_T$. An estimator is called *unbiased* if the mean of the estimator is equal to the parameter that is being estimated. Determine if the ML estimator of b is unbiased.

Problem: Suppose that X & Y are continuous-type and independent. Show that $P(X=Y)=0$.

Solution: Recall that $P((X,Y)\in A) = \iint_A f_{X,Y}(u,v)\,du\,dv$. Here $A = \{(x,y)\in\mathbb{R}^2 : x=y\}$ is a line & has 1 dimension. Hence $P((X,Y)\in A)=0$.

Another way to see this:

$$P(X=Y) = \lim_{\varepsilon\to\infty} P(|X-Y|<\varepsilon) = \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\int_{u-\varepsilon}^{u+\varepsilon} f_{X,Y}(u,v)\,dv\,du$$

$$= \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\int_{v-\varepsilon}^{u+\varepsilon} f_X(u) f_Y(v)\,dv\,du = \int_{-\infty}^{+\infty} f_X(u) \lim_{\varepsilon\to 0}\int_{u-\varepsilon}^{u+\varepsilon} f_Y(v)\,dv\,du = 0.$$

Today: joint pdf of functions of random variables

① Transformation of pdfs under a linear mapping

Suppose that X and Y are jointly-continuous with pdf $f_{X,Y}$. Suppose that $W = aX+bY$ and $Z = cX+dY$ for some constants a,b,c,d. Our goal is to find $f_{W,Z}$. We can write:

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If A is invertibe: if A is invertible, we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A^{-1}\begin{pmatrix} W \\ Z \end{pmatrix}$$

Suppose (X,Y) is in u-v plane, and (W,Z) is in α-β plane. We have,

$$P(W \leq e, Z \leq f) = P\left(A\begin{pmatrix} X \\ Y \end{pmatrix} \leq \begin{pmatrix} e \\ f \end{pmatrix} \right)$$

$$= \iint_{A\binom{u}{v} \leq \binom{e}{f}} f_{X,Y}(u,v)\, dv\, du$$

$$= \int_{-\infty}^{e}\int_{-\infty}^{f} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \frac{d\beta\, d\alpha}{|\det(A)|}$$

change of variable $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$

we have $d\alpha\, d\beta = |\det(A)|\, du\, dv$

Proposition: Suppose that $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$, where $\begin{pmatrix} X \\ Y \end{pmatrix}$ has pdf $f_{X,Y}$, and A is a matrix with $\det(A) \neq 0$. Then $\begin{pmatrix} W \\ Z \end{pmatrix}$ has joint pdf given by

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

If A is not invertible: Then W and Z are linearly related to each other, i.e., $W = \frac{a}{c} Z$.

In this case, it only makes sense to talk about pdf of W.