Lecture 32 - 11/09

Wednesday, November 09, 2022 10:43 AM

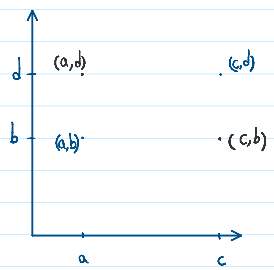
Review

- Independence of random variables:
  - X & Y are independent if & only if $F_{X,Y}(u,v) = F_X(u)F_Y(v)$ $\forall u,v \in \mathbb{R}$
  - If X & Y are discrete type, they are independent if & only if $P_{X,Y}(u,v) = P_X(u)P_Y(v)$ $\forall u,v \in \mathbb{R}$
  - If X & Y are jointly continuous, they are independent if & only if $f_{X,Y}(u,v) = f_X(u)f_Y(v)$ $\forall u,v \in \mathbb{R}$
- Determining from a joint pdf whether independence holds

To ensure independence: for every $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$ only depends of v, i.e., $f_{Y|X}(v|u) = f_Y(v)$.

To roll-out independence: if X & Y are independent then support of $f_{X,Y}$ should satisfy swap property, i.e.,

$$\begin{matrix} (a,b) \in \text{support of } f_{X,Y} \\ (c,d) \in \text{support of } f_{X,Y} \end{matrix} \Rightarrow \begin{matrix} (a,d) \in \text{support of } f_{X,Y} \\ (b,c) \in \text{support of } f_{X,Y} \end{matrix}$$



- Function of pair of jointly continuous random variables:

$Z = g(X,Y)$ & the joint pdf of (X,Y) is given by f_{XY}. Find distribution of Z:

Step 1: identify type of Z (continuous-type or discrete-type) and support of Z (c s.t. $f_Z(c) > 0$)

→ if Z is continuous-type: Step 2: find its cdf. $F_Z(c) = P(Z \leq c) = P(g(X,Y) \leq c) = \iint_{(u,v):\, g(u,v) \leq c} f_{X,Y}(u,v)\,du\,dv$

Step 3: take derivative of $F_Z(c)$ to derive pdf of Z: $f_Z(c) = \frac{\delta F_Z(c)}{\delta c}$

→ if Z is discrete-type: Step 2: calculate pmf of Z: $P(Z=k) = P(g(X,Y)=k) = \iint_{(u,v):\, g(u,v)=k} f_{X,Y}(u,v)\,du\,dv$

- Sum of random variables:

① Suppose that X & Y are discrete-type, and integer valued. Let $S = X+Y$.

$$P_S(k) = P(S=k) = \sum_j P(X=j, Y=k-j) = \sum_j P_{X,Y}(j, k-j)$$

If X and Y are independent, then

$$\boxed{P_S(k) = P(S=k) = \sum_j P_X(j) P_Y(k-j) =: P_X * P_Y(k) \quad \text{if } X \text{ \& } Y \text{ are independent \& } S = X+Y}$$

② Suppose that X and Y are jointly continuous. Let $S = X+Y$.

$$f_S(c) = \int_{-\infty}^{+\infty} f_{X,Y}(u, c-u)\,du = \int_{-\infty}^{+\infty} f_{X,Y}(c-v, v)\,dv$$

If X and Y are independent.

$$\boxed{f_S(c) = \int_{-\infty}^{+\infty} f_X(u) f_Y(c-u)\,du = \int_{-\infty}^{+\infty} f_X(c-v) f_Y(v)\,dv = f_X * f_Y(c) \quad \text{if } X \text{ \& } Y \text{ are independent \& } S = X+Y}$$

Example 4.6.1 Suppose $W = \max(X, Y)$, where X and Y are independent, continuous-type random variables. Express f_W in terms of f_X and f_Y. (Note: This example complements the analysis of the minimum of two random variables, discussed in Section 3.9 in connection with failure rate functions.)

Example 4.6.2 Suppose X and Y are jointly continuous-type random variables. Let $R = \sqrt{X^2 + Y^2}$, so R is the distance of the random point (X, Y) from the origin. Express f_R in terms of $f_{X,Y}$.

Example 4.6.3 Buffon's needle problem Suppose a needle of unit length is thrown at random onto a large grid of lines with unit spacing. Find the probability the needle, after it comes to rest, intersects a grid line.