Review Lecture

Friday, November 04, 2022 9:51 AM

Week 6:

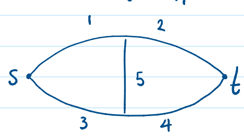
① Union bound: $\begin{cases} P(A \cup B) \leq P(A) + P(B), & \text{for any } A, B \in \mathcal{F} \\ P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} P(A_i), & \text{for any } A_1, \ldots, A_n \in \mathcal{F} \end{cases}$

② Network outage probability:

. We are sending packets from source node to terminal node using an underlying s-t network.

. link i in the s-t network fails with probability $p_i = P(F_i)$, independent of everything else.

. A network outage happens if at least one link fails over each path from s to t.



$\text{Outage} = F_1F_3 \cup F_2F_4 \cup F_1F_4F_5 \cup F_2F_3F_5$

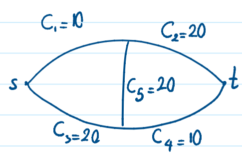
Goal: $\begin{cases} \text{(i) exactly calculate } P(\text{Outage}) = P(F_5)P(F \mid F_5) + P(F_5^c)P(F \mid F_5^c) \\ \text{(ii) use union bound to upperbound } P(\text{Outage}) \leq P(F_1F_3) + P(F_2F_4) + P(F_1F_4F_5) + P(F_3F_2F_5) \end{cases}$

③ Distribution of capacity of flow network

. There is a capacity assigned to each link:

. if link i fails, it cannot pass any packets.

. if link i works, it can pass packets upto its capacity C_i.



Goal: pmf of capacity of network, i.e., pmf of number of packets that can reach terminal node, from source node.

④ Cummulative distribution function:

Def: $F_X(c) = \mathbb{P}(X \leq c) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq c\})$, for any $c > 0$.

main properties: F is cdf of some random variable if & only if:

F.1. increasing: $a < b \Rightarrow F(a) < F(b)$

F.2. $\lim_{c \to \infty} F(c) = 1$, $\lim_{c \to -\infty} F(c) = 0$

F.3. F is right continuous, i.e., $\lim_{u \to c^+} F(u) = F(c)$

other properties:

(i) $F_X(c-) = \lim_{u \to c^-} F_X(u) = P(X < c)$

(ii) $\Delta F_X(c) = F_X(c) - F_X(c-) = P(X = c)$

(iii) For any $a < b$, $F_X(b) - F_X(a) = P(a < X \leq b)$, in particular $0 \leq F(a) \leq 1$.

⑤ Continuous random variable:

Def. X is continuous-type random variable if $F_X(c) = \int_{-\infty}^{c} f_X(u)\,du$; f_X is the probability density function of X.

- Support of f_X is the set u for which $f_X(u) > 0$

main properties: f is pdf of some random variable if & only if:

- $f_X(u) \geq 0$ for all $u \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f_X(u)\,dx = 1 = \lim_{c \to \infty} F_X(c)$

other properties.

- If f_X is continuous at u, then $F_X'(u) = f_X(u)$
- $P(X=u) = 0$ for all continuous random variables & any $u \in \mathbb{R}$.

⑥ Uniform distribution:

Def. X is uniformly distributed over $[a,b]$, $X \sim Unif([a,b])$

$$f_X(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{O.W} \end{cases}, \quad \mu_X = E[X] = \frac{b+a}{2}, \quad var(X) = \frac{(b-a)^2}{12}$$

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week 7.

① exponential distribution:

Def. T has exponential distribution with parameter $\lambda > 0$, $T \sim exp(\lambda)$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{O.W} \end{cases}$$

- cdf: $P(T \leq t) = F_T(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & \text{O.W.} \end{cases}$. complimentary cdf: $P(T > t) = F_T^c(t) = \begin{cases} e^{-\lambda t} & t \geq 0 \\ 0 & \text{O.W.} \end{cases}$
- Mean & Variance: $E[T] = \frac{1}{\lambda}$, $Var(T) = \frac{1}{\lambda^2}$
- Memoryless property:

$$P(T > t+s \mid T > s) = P(T > t)$$

② relation between geometric distribution and exponential distribution:

Let X_h denote a geometric random variable with parameter $p_h = \lambda h$ for all $h > 0$. Let $T_h = hX_h$. We have

$$P(T_h > t) \longrightarrow P(T > t) \quad \text{as } h \to 0$$

where T is a exponentially distributed random variable with parameter $\lambda > 0$.

③ Poisson process:

Def: A Poisson process with rate $\lambda > 0$ is a conting process $N = (N_t : t \geq 0)$ that satisfies the followings

(N.1) It has independent incremant property: $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$, then $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \dots, N_{t_n} - N_{t_{n-1}}$

(N.1) It has independent increament property: $0\leq t_1\leq t_2\leq\cdots\leq t_k$, then $N_{t_2}-N_{t_1}, N_{t_3}-N_{t_2}, \ldots N_{t_n}-N_{t_{n-1}}$ are independent

(N2) For any $t>s$, N_t-N_s has Poisson distribution with parameter $\lambda(t-s)$

(N3) $N_0=0$.

. Some related random variables.

(i) U_n = inter arrival time between (n-1)'th and n'th arrival, U_n has exponential distribution with parameter λ.

(ii) T_r = time till the r'th arrival, T_r has Erlang distribution with parameters (r,λ)

$$f_{T_r}(t)=\begin{cases}\dfrac{\lambda e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!}, & t\geq 0\\ 0, & t<0\end{cases}$$

$$F_{T_r}(t)=1-\sum_{i=0}^{r-1}\frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

Notice that $\int_{-\infty}^{+\infty} f_{T_r}(u)\,du=1$, since it is a pdf.

. Relation between above random variables:

(i) $N_t=\sum_{r=1}^{\infty} 1_{\{T_r\leq t\}}$ where $1_{\{\cdot\}}$ is indicator function.

$$1_A=\begin{cases}1 & \text{if } A \text{ is true}\\ 0 & \text{if } A \text{ is false}\end{cases}$$

(ii) $T_r=U_1+U_2+\cdots+U_r$

(iii) $T_r=\min\{t: N_t\geq r\}$

Proposition: Let N be a random counting process and let $\lambda>0$. The following are equivalent.

(i) N is a Poisson process with parameter λ.

(ii) The intercount (inter arrival) times $U_1, U_2, \ldots, U_n$ are mutually independent and exponentially distributed random variables.

④ Scaling of random variables:

$Y=aX+b \Rightarrow f_Y(u)=\frac{1}{a}f_X\left(\frac{u-b}{a}\right)$

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week 8

① Guassian distribution.

Def: X has guassian (normal) distribution, $X\sim N(\mu,\sigma^2)$ if

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Def: X has guassian (normal) distribution, $X \sim N(\mu, \sigma^2)$ if

$$f_X(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right), \quad u \in \mathbb{R}$$

μ. mean $\quad \sigma^2$. variance

- Standardized version of X, $\tilde{X} = \frac{X-\mu}{\sigma}$ is distributed as $N(0,1)$.
  - $F_{\tilde{X}}(a) = P(\tilde{X} \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{u^2}{2}\right) du = \Phi(a)$
  - $F^c_{\tilde{X}}(a) = P(\tilde{X} > a) = \int_{a}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = 1 - \Phi(a) = Q(a)$
  - Notice that $Q(a) = \Phi(-a)$ because of symmetry!
- We can write:

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) - P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X > b) = P\left(\frac{X-\mu}{\sigma} > \frac{b-\mu}{\sigma}\right) = Q\left(\frac{b-\mu}{\sigma}\right) = \Phi\left(-\frac{b-\mu}{\sigma}\right)$$

② Guassian approximation with error correction:

Let $S \sim Bi(n,p)$ denote a random variable with Binomial distribution. Let $X \sim N(\mu, \sigma^2)$ denote a Guassian random variable with same mean and variance as S, i.e., $\mu = np$, $\sigma^2 = np(1-p)$

$$\left.\begin{aligned} P(S \leq k) &\approx P(X \leq k+0.5) \\ P(S > k) &\approx P(X \geq k+0.5) \end{aligned}\right\} \text{ sum of these two should be one}$$

$$P(S \geq k) \approx P(X \geq k - 0.5)$$

③ ML parameter estimation for continuous-type random variables

(i) pdf of random variable X belongs to a family of parametrized distributions: f_θ

(ii) We are given an observation $X = u$.

(iii) We want to find the parameter that maximizes the pdf at u, i.e., $\hat{\theta}_{ML} = \arg\max_{\theta} f_\theta(u)$

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week 9:

① The distribution of a function of a random variable:

Suppose that $Y = g(X)$. X is continuous type. We want to find distribution of Y.

Step 1: identify type of Y (continuous-type or discrete-type) and support of Y (u s.t. $f_Y(u) > 0$)

→ if Y is continuous-type.

Step 2: find its cdf. $F_Y(c) = P(Y \leq c) = P(g(X) \leq c) = \int_{u:\, g(u) \leq c} f_X(u)\, du$

Step 3: take derivative of $F_Y(c)$ to derive pdf of Y.

$$f_Y(c) = \frac{\delta F_Y(c)}{\delta c}$$

Step 3: take derivative of $F_Y(c)$ to derive pdf of Y:

$$f_Y(c) = \frac{\delta F_Y(c)}{\delta c}$$

→ if Y is discrete-type: Step 2: calculate pmf of Y. $P(Y=k) = P(g(X)=k) = \int_{u:\,g(u)=k} f_X(u)\,du.$

• Important example: X is continuous-type with CDF F_X. $Y = F_X(X)$ is uniformly distributed between 0 and 1.

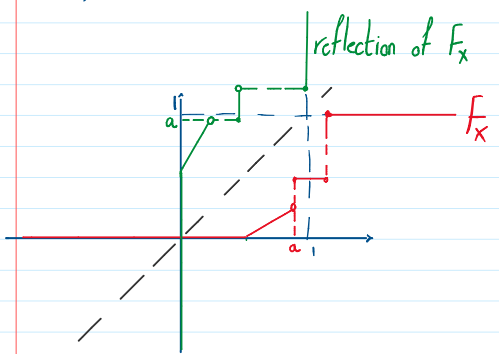
② Generating variable with specified distribution:

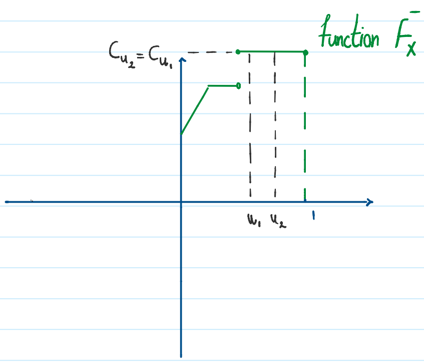
. Given $U \sim unif([0,1])$ and a cdf F, find function g such that the cdf of $X = g(U)$ is F, i.e., for any $c \in \mathbb{R}$ $F_X(c) = F(c)$.

Answer: $g(u) = F^{-1}(u) = \min\{c : F(c) \geq u\}$.

(a) if possible calculate F^{-1}

(b) if it is hard use reflection & then make it into a function.





③ Hypothesis testing for continuous type observation.

(i) The system generates continuous-type random variable X

. If the system is in state H_0, the pdf of X is given by f_0

. If the system is in state H_1, the pdf of X is given by f_1

(ii) We observe a realization of X, i.e., we observe $X = u$.

(iii) We guess the state of system, using our observation, based on the decision rule.

We are only interested in threshold decision rule that compare $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$ with threshold τ.

ML: $\tau = 1$, MAP: $\tau = \frac{\pi_0}{\pi_1}$

Week 10:

① joint cdf.

Def. For two random variables X and Y that are defined over the same probability space:

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = P\{\omega \in \Omega : X(\omega) \leq u, Y(\omega) \leq v\}$$

main properties: A function F is a joint cdf if and only if:

JF.1: $0 \leq F(u,v) \leq 1$ for any $(u,v) \in \mathbb{R}^2$

JF.2: $F(u,v)$ is nondecreasing in u and nondecreasing in v.

JF.2: $F(u,v)$ is nondecreasing in u and nondecreasing in v.

JF.3: $F(u,v)$ is right-continuous in u and right-continuous in v.

JF.4: $\lim_{u\to-\infty} F(u,v)=0$ and $\lim_{v\to-\infty} F(u,v)=0$

JF.5: $\lim_{u\to\infty}\lim_{v\to\infty} F(u,v)=1$

JF.6. For any $a<b$ and $c<d$, $F(b,d)-F(a,d)-F(b,c)+F(a,c)\geq 0$

other properties:

- $P(a<X\leq b,\ c<Y\leq d)=F_{X,Y}(b,d)-F_{X,Y}(a,d)-F_{X,Y}(b,c)+F_{X,Y}(a,c)$
- $\lim_{u\to+\infty} F_{X,Y}(u,v)=F_Y(v)$, $\lim_{v\to+\infty} F_{X,Y}(u,v)=F_X(u)$

② joint pmf:

. If X and Y are discrete type, their joint pmf is defined as

$$P_{X,Y}(u,v)=P(X=u\ Y=v)=P(\{w\in\Omega: X(w)=u,\ Y(w)=v\})$$

there exists $\{u_1,u_2,\dots\}$ and $\{v_1,v_2,\dots\}$ such that $P_{X,Y}(u,v)=0$ if $u\notin\{u_1,u_2,\dots\}$ or $v\notin\{v_1,v_2,\dots\}$.

. Conditional probability

$$P_{Y|X}(v|u)=P(Y=v|X=u)=\frac{P(X=u,Y=v)}{P(X=u)}=\frac{P_{X,Y}(u,v)}{P_X(u)}$$

. Relation between marginal pmf and joint pmf

$$P_X(u)=\sum_i P_{X,Y}(u,v_i),\quad P_Y(v)=\sum_i P_{X,Y}(u_i,v)$$

③ joint pdf:

We say random variables X and Y are jointly continuous if

$$F_{X,Y}(u_0,v_0)=\int_{-\infty}^{u_0}\int_{-\infty}^{v_0} f_{X,Y}(u,v)\,dv\,du$$

the function $f_{X,Y}$ is called the joint pdf.

. For any region $A\subset\mathbb{R}^2$, we have

$$P((X,Y)\in A)=\iint_A f_{X,Y}(u,v)\,du\,dv$$

. Given a function $g:\mathbb{R}^2\to\mathbb{R}$, we can use LOTUS to calculate $E[g(X,Y)]$:

$$E[g(X,Y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\,f_{X,Y}(u,v)\,du\,dv$$

. Def: conditional pdf of Y given X, denoted by $f_{Y|X}$ is defined as

$$f_{Y|X}(v_0|u_0) = \begin{cases} \dfrac{f_{X,Y}(u_0,v_0)}{f_X(u_0)} & \text{if } f_X(u_0)>0 \\ \text{undefined} & \text{o.w.} \end{cases} \quad , \text{ for any } (u_0,v_0)\in\mathbb{R}^2$$

$P(Y\in A|X=u_0) = \int_A f_{Y|X}(v|u_0)\,dv$ $\rightsquigarrow$ conditional probability of Y given $X=u$

$E[Y|X=u_0] = \int_{-\infty}^{+\infty} v\cdot f_{Y|X}(v|u_0)\,dv$ $\rightsquigarrow$ conditional expectation of Y given $X=u$

. Given $Z=g(X,Y)$, we find the distribution of Z by taking the following steps:

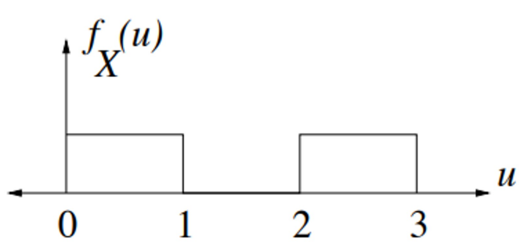
Step 1: identify type of Z (continuous-type or discrete-type) and support of Z (c s.t. $f_Z(c)>0$)

→ if Z is continuous-type: Step 2: find its cdf. $F_Z(c) = P(Z\le c) = P(g(X,Y)\le c) = \iint_{(u,v):\, g(u,v)\le c} f_{X,Y}(u,v)\,du\,dv$

Step 3: take derivative of $F_Z(c)$ to derive pdf of Z: $f_Z(c) = \dfrac{\delta F_Z(c)}{\delta c}$

→ if Z is discrete-type: Step 2: calculate pmf of Z: $P(Z=k) = P(g(X,Y)=k)$

1. **[24 points]** (6 points) Suppose X is a random variable with the pdf shown. (Fall 2014)



(a) (6 points) Carefully *sketch* and label the CDF of X.

(b) (6 points) Carefully *sketch* and label the pdf of Y, where $Y = 2X + 1$.

(c) (6 points) Find $E[X]$. Simplify as much as possible. Show your work or explain your reasoning.

(d) (6 points) Find $P\{\ln(X) \geq 1\}$. (Hint: $e = \exp(1) \approx 2.72$.)

2. **[14 points]** Suppose that the lifetime of a battery produced by company EBunny can be modeled by a Gaussian random variable, with mean 300 days and variance 100 (days)^2. (Fall 2014)

(a) (7 points) A buyer is interested in purchasing the batteries, but requires that at least 99% of the batteries delivered have lifespan of at least 275 days. Do the EBunny batteries meet the requirement? Justify your answer. (Note: A table of the Φ function is on the last page of the exam booklet.)

meet the requirement? Justify your answer. (Note: A table of the Φ function is on the last page of the exam booklet.)

(b) (7 points) Let Y be a binomial random variable with parameters n and p such that Y has the same expected value and variance as the Gaussian variable in the original problem statement. Find n and p.

3. **[12 points]** Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. (Fall 2014)

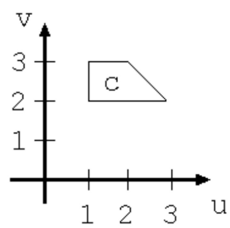
(a) (6 points) Express $E[N(t)N(t+s)],\ s, t > 0$ as a function of λ, s, and t.

(b) (6 points) Let $\lambda = 2$ arrivals/hour and assume that the Poisson process models the arrival of customers into a post office. Find the probability of the following event, which involves three conditions:

(Three customers arrive between 1 and 3pm,
one customer arrives between 2 and 3pm,
and one customer arrives between 2 and 4pm.)

3. **[20 points]** Let X and Y be two random variables with joint pdf (Spring 2015)

$$f_{X,Y}(u,v) = \begin{cases} c & 1 \leq u \leq 2,\ 2 \leq v \leq 3, \\ c & 2 \leq u \leq 3,\ 2 \leq v \leq 5-u, \\ 0 & \text{else.} \end{cases},$$



where c is a constant.

(a) Determine the value of the constant c for $f_{X,Y}$ to be a valid joint pdf.

(b) Determine the marginal pdf of Y, $f_Y(v)$, for all v.

(c) Determine the conditional pdf $f_{X|Y}(u|v)$ for all u, v.