Lecture 30 - 11/02

Wednesday, November 02, 2022 2:04 AM

Review:

Independence of random variables:

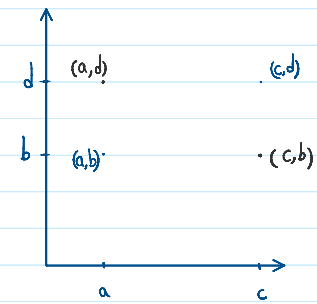
- X & Y are independent if & only if $F_{X,Y}(u,v) = F_X(u)F_Y(v)$ $\forall u,v \in \mathbb{R}$
- If X & Y are discrete type, they are independent if & only if $p_{X,Y}(u,v) = p_X(u)\,p_Y(v)$ $\forall u,v \in \mathbb{R}$
- If X & Y are jointly continuous, they are independent if & only if $f_{X,Y}(u,v) = f_X(u)\,f_Y(v)$ $\forall u,v \in \mathbb{R}$

- Determining from a joint pdf whether independence holds

To ensure independence: for every $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$ only depends of v, i.e., $f_{Y|X}(v|u) = f_Y(v)$.

To roll-out independence: if X & Y are independent then support of $f_{X,Y}$ should satisfy swap property, i.e.,

$$\left.\begin{array}{l}(a,b) \in \text{support of } f_{X,Y} \\ (c,d) \in \text{support of } f_{X,Y}\end{array}\right. \Rightarrow \begin{array}{l}(a,d) \in \text{support of } f_{X,Y} \\ (b,c) \in \text{support of } f_{X,Y}\end{array}$$



Today: ① Distribution of sum of two random variables

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① Distribution of sum of two random variables

Recall that given $Z = g(X,Y)$, we find the distribution of Z by taking the following steps:

Step 1: identify type of Z (continuous-type or discrete-type) and support of Z (c s.t. $f_Z(c) > 0$)

- if Z is continuous-type: Step 2: find its cdf. $F_Z(c) = P(Z \leq c) = P(g(X,Y) \leq c) = \iint_{(u,v):\, g(u,v) \leq c} f_{X,Y}(u,v)\,du\,dv$

  Step 3: take derivative of $F_Z(c)$ to derive pdf of Z:
  $$f_Z(c) = \frac{\delta F_Z(c)}{\delta c}$$

- if Z is discrete-type: Step 2: calculate pmf of Z. $P(Z = k) = P(g(X,Y) = k)$

Here, we are interested in case where $g(X,Y) = X + Y$.

→ If Z is discrete-type: Step 2: calculate pmf of Z: $P(Z=k) = P(g(X,Y)=k)$

Here, we are interested in case where $g(X,Y) = X+Y$.

. Sum of integer-valued random variables.

Suppose that $S = X+Y$, and joint pmf of X and Y is $P_{X,Y}(u,v)$ for all $u,v \in \mathbb{R}$.

Suppose that X & Y are integer-valued, i.e., $P_X(u) = 0$ and $P_Y(v) = 0$ for any $u,v \notin \{\dots,-2,-1,0,1,2,\dots\}$

For any $k \in \{\dots,-2,-1,0,1,2,\dots\}$ we have:

$$P_S(k) = P(S=k) = P(X+Y=k) = \sum_j P(X=j, Y=k-j) = \sum_j P_{X,Y}(j,k-j)$$

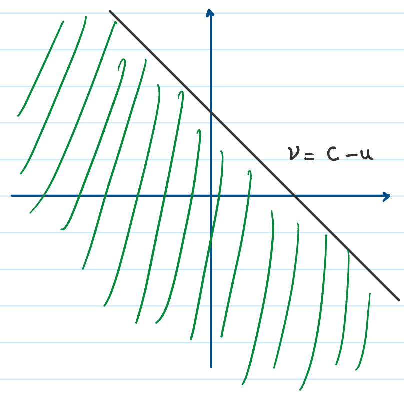
If X and Y are independent, then

$$\boxed{P_S(k) = P(S=k) = \sum_j P_X(j) P_Y(k-j) =: P_X * P_Y(k) \quad \text{if } X \text{ \& } Y \text{ are independent \& } S = X+Y}$$

where '$*$' denote the convolution operator.

. Sum of jointly continuous random variables.

Suppose that $S = X+Y$ and X and Y are jointly continuous with joint pdf $f_{X,Y}(u,v)$, for $(u,v) \in \mathbb{R}^2$. We will find $F_S(c)$ and then take derivative.



Step 2: $F_S(c) = P(S \leq c) = P(X+Y \leq c) = \iint_{(u,v):\, u+v \leq c} f_{X,Y}(u,v)\,dv\,du = \int_{-\infty}^{+\infty} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\,dv\,du$

Step 3: $f_S(c) = \frac{dF_S(c)}{dc} = \int_{-\infty}^{+\infty} \frac{d}{dc} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\,dv\,du = \int_{-\infty}^{+\infty} f_{X,Y}(u, c-u)\,du = \int_{-\infty}^{+\infty} f_{X,Y}(c-v, v)\,dv$

The integral in above is integral of $f_{X,Y}$ over line $u+v=c$, and it is the law of total probability for having $X+Y=c$.

If X and Y are independent.

$$\boxed{f_S(c) = \int_{-\infty}^{+\infty} f_X(u) f_Y(c-u)\,du = \int_{-\infty}^{+\infty} f_X(c-v) f_Y(v)\,dv = f_X * f_Y(c) \quad \text{if } X \text{ \& } Y \text{ are independent \& } S = X+Y}$$

where '$*$' denote the convolution operator.

4.15. **[Joint Densities and functions of two random variables]**
Let X and Y have joint pdf

$$f_{X,Y}(u,v) = \begin{cases} A\left(1 - \sqrt{u^2+v^2}\right) & u^2+v^2 < 1 \\ 0 & \text{else.} \end{cases}$$

Hint: Use of polar coordinates is useful for all parts of this problem.

(a) Find the value of A.

(b) Let $Z = X^2 + Y^2$. Find the pdf of the random variable Z.

(c) Find $E[Z^5]$ using LOTUS for joint pdfs: $E[g(X,Y)] = \int\int_{\mathbb{R}^2} g(u,v) f_{X,Y}(u,v) dudv$.

4.13. **[Joint densities]**
X and Y are two random variables with the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} A(1 - |u-v|) & 0 < u < 1, 0 < v < 1 \\ 0 & else \end{cases}$$

(a) Find A.

(b) Find marginal pdfs for X and Y.

(c) Find $P\{X > Y\}$.

(d) Find $P(X + Y < 1 | X > 1/2)$.