Independence of random variables.

- \( X \) & \( Y \) are independent if & only if \( F_{X,Y}(u,v) = F_X(u)F_Y(v) \) \( \forall u,v \in \mathbb{R} \)

- if \( X \) & \( Y \) are discrete type, they are independent if & only if \( P_{X,Y}(u,v) = P_X(u)P_Y(v) \) \( \forall u,v \in \mathbb{R} \)

- if \( X \) & \( Y \) are jointly continuous, they are independent if & only if \( f_{X,Y}(u,v) = f_X(u)f_Y(v) \) \( \forall u,v \in \mathbb{R} \)

Determining from a joint pdf whether independence holds.

To ensure independence: for every \( u \in \mathbb{R} \), either \( f_{X,Y}(u,v) \) or \( f_{Y,X}(v,u) \) only depends on \( v \), i.e., \( f_{X,Y}(u,v) = f_Y(v) \).

To roll out independence: if \( X \) & \( Y \) are independent then support of \( f_{X,Y} \) should satisfy swap property, i.e.,

\[(a,b) \in \text{support of } f_{X,Y} \quad \Rightarrow \quad (a,d) \in \text{support of } f_{X,Y} \quad \text{or} \quad (c,d) \in \text{support of } f_{X,Y} \]

\[(c,d) \in \text{support of } f_{X,Y} \quad \Rightarrow \quad (b,c) \in \text{support of } f_{X,Y} \]

\[\text{and} \quad (a,d) \in \text{support of } f_{X,Y} \quad \Rightarrow \quad (c,b) \in \text{support of } f_{X,Y} \]

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Today. 0 Distribution of sum of two random variables

0 Distribution of sum of two random variables

Recall that given \( Z = g(X,Y) \), we find the distribution of \( Z \) by taking the following steps.

Step 1: Identify type of \( Z \) (continuous type or discrete type) and support of \( Z \) (c.s.t. \( f_Z(c) > 0 \))

- If \( Z \) is continuous type, find its cdf. \( F_Z(c) = P(Z \leq c) = \int_{\mathbb{R}^2} f_{X,Y}(u,v) \, du \, dv \) with \( g(u,v) \leq c \)

- Take derivative of \( F_Z(c) \) to derive pdf of \( Z \):

\[ f_Z(c) = \frac{dF_Z(c)}{dc} \]

- If \( Z \) is discrete type, step 2: calculate pmf of \( Z \). \( P(Z = k) = P(g(X,Y) = k) \)

Here, we are interested in case where \( g(X,Y) = X + Y \).
Here, we are interested in case where \( g(X,Y) = X + Y \).

### Sum of integer-valued random variables.

Suppose that \( S = X + Y \), and joint pmf of \( X \) and \( Y \) is \( p_{XY}(x,y) \) for all \( x, y \in \mathbb{R}^+ \).

Suppose that \( X \) and \( Y \) are integer-valued, i.e., \( p_X(x) = 0 \) and \( p_Y(y) = 0 \) for any \( x, y \notin \{-2,-1,0,1,2,\ldots\} \).

For any \( k \in \{-2,-1,0,1,2,\ldots\} \) we have:

\[
P_S(k) = P(S = k) = P(X + Y = k) = \sum_{j} P(X = j, Y = k - j) = \sum_{j} p_{XY}(j, k - j)
\]

If \( X \) and \( Y \) are independent, then:

\[
P_S(k) = P(S = k) = \sum_{j} p_X(j) p_Y(k - j) = p_X \ast p_Y(k) \quad \text{if } X \text{ and } Y \text{ are independent} \& \ S = X + Y
\]

where \( \ast \) denote the convolution operator.

### Sum of jointly continuous random variables.

Suppose that \( S = X + Y \) and \( X \) and \( Y \) are jointly continuous with joint pdf \( f_{XY}(x,y) \), for \( (x,y) \in \mathbb{R}^+ \). We will find \( F_S(c) \) and then take derivative.

**Step 2:**

\[
F_S(c) = P(S \leq c) = P(X + Y \leq c) = \int_{-\infty}^{c} \int_{-\infty}^{c-x} f_{XY}(x,y) \, dx \, dy = \int_{-\infty}^{c} f_X(c-x) \, dx
\]

**Step 3:**

\[
\frac{dF_S(c)}{dc} = \int_{-\infty}^{c} f_X(c-x) \, dx = \int_{-\infty}^{\infty} f_X(u) \, du = 1
\]

The integral in above is integral of \( f_X \) over line \( u + v = c \), and it is the law of total probability for having \( X + Y = c \).

If \( X \) and \( Y \) are independent:

\[
f_S(c) = \int_{-\infty}^{\infty} f_X(x) f_Y(c-x) \, dx = \int_{-\infty}^{\infty} f_X(c-u) f_Y(u) \, du = f_X \ast f_Y(c) \quad \text{if } X \text{ and } Y \text{ are independent} \& \ S = X + Y
\]

where \( \ast \) denote the convolution operator.
4.15. **[Joint Densities and functions of two random variables]**

Let X and Y have joint pdf

\[ f_{X,Y}(u, v) = \begin{cases} 
A \left(1 - \sqrt{u^2 + v^2}\right) & u^2 + v^2 < 1 \\
0 & \text{else.} \end{cases} \]

Hint: Use of polar coordinates is useful for all parts of this problem.

(a) Find the value of A.

(b) Let \( Z = X^2 + Y^2 \). Find the pdf of the random variable Z.

(c) Find \( E[Z^5] \) using LOTUS for joint pdfs: \( E[g(X, Y)] = \int \int_{\mathbb{R}^2} g(u, v) f_{X,Y}(u, v) dudv \).

4.13. **[Joint densities]**

X and Y are two random variables with the following joint pdf:

\[ f_{X,Y}(u, v) = \begin{cases} 
A(1 - |u - v|) & 0 < u < 1, 0 < v < 1 \\
0 & \text{else} \end{cases} \]

(a) Find A.

(b) Find marginal pdfs for X and Y.

(c) Find \( P\{X > Y\} \).

(d) Find \( P(X + Y < 1 | X > 1/2) \).