Lecture 3 - 8/26

Review:

1. Axioms of probability: probability space $(\Omega, \mathcal{F}, P)$

   1.1 Axioms of events

   1.2 Axioms of probability

1.1 Axioms of events:

E.1: $\Omega \in \mathcal{F}$

E.2: if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

E.3: if $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i=0}^{\infty} A_i \in \mathcal{F}$

1.2 Axioms of Probability:

P.1: $P(A) \geq 0$, $\forall A \in \mathcal{F}$

P.2: $P(\Omega) = 1$

P.3: $A_1, A_2, \ldots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, mutually exclusive

$$P\left(\bigcup_{i=0}^{\infty} A_i\right) = \sum_{i=0}^{\infty} P(A_i)$$

2. Counting size of sets.

2.1 Principle of counting: We have a giant task which consist of two subtasks. Giant task will be done if both subtasks are accomplished:

- there are m ways to do subtask 1

. there are n ways to do subtask 2

$\Rightarrow$ There are mn ways to do subtask 1 AND 2

(2.2) Principle of sum: We have task that can be done in two different ways. The task is done if either we do it the first way or the other.

. there are m ways to do the task one way.

. there are n ways to do the task the other way.

$\Rightarrow$ There are $m+n$ ways to do the task using first method OR the second method.

(3.3) Principle of overcounting: If in the process of counting # of ways to do a task, each method has been counted K times, the derived # should be divided by K to get # of different ways to accomplish the task.

Today:

1. Counting size of sets
2. Probability with equal outcomes
3. Karnaugh map with 3 sets.

① Counting size of sets

e.1: $\Omega = \{1,2,3,4,5\}$

(a) # of subset:

The task: constructing $A \subset \Omega$.

Subtasks: picking elements

$$\underset{1\in A \text{ or not}}{\underline{2}} \times \underset{2\in A \text{ or not}}{\underline{2}} \times \cdots \times \underset{5\in A \text{ or not}}{\underline{2}} = 2^5$$

(b) # of subsets of size 3:

The task: constructing $A \in \Omega$, $|A|=3$

Suppose order matters: $\underset{\text{Pick first member}}{\underline{5}} \times \underset{\text{2nd element}}{\underline{4}} \times \underset{\text{3rd elem.}}{\underline{3}}$

# of times we counted abc: $3!$ $\rightarrow$ $\frac{5\times 4\times 3}{3!} = \binom{5}{3}$

e.2: # of words with A, B, C, D, E

3 letters, repetition allowed: $\underset{\substack{\text{choose}\\\text{1st word}}}{\underline{5}} \times \underset{\substack{\text{choose}\\\text{2nd word}}}{\underline{5}} \times \underset{\substack{\text{choose}\\\text{3rd word}}}{\underline{5}} = 5^3$

4 letters, repetition not allowed. $\underset{\substack{\text{choose}\\\text{1st word}}}{\underline{5}} \times \underset{\substack{\text{choose}\\\text{2nd word}}}{\underline{4}} \times \underset{\substack{\text{choose}\\\text{3rd word}}}{\underline{3}} \times \underset{\substack{\text{choose}\\\text{4th word}}}{\underline{2}} = 5!$

3 letters, repetition not allowed, A and B cannot come together

A comes. $\underset{\text{location}}{\underline{3}} \times \underset{\text{filling remaining}}{\underline{3\times 2}} = 3\times 3!$

A comes : $\underset{\text{location of } A}{\underline{3}} \times \underset{\text{filling remaining locations with } \{C,D,E\}}{\underline{3\times 2}} = 3\times 3!$

B comes : $\underset{\text{location of } B}{\underline{3}} \times \underset{\text{filling remaining locations with } \{C,D,E\}}{\underline{3\times 2}} = 3\times 3!$

Neither come : $\underset{\text{filling locations with } \{C,D,E\}}{\underline{3\times 2\times 1}} = 3!$

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def. $\binom{n}{k}$: # of ways to pick k items out of n differen items
= # of subsets of size k in a set with n elements.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$n(n-1)\cdots(n-k+1)$ ⇝ assume order matters

$k!$ ⇝ # of times each combinations has been counted

Obs. $(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n \text{ times}}$

# of ways to construct $a^k b^{n-k}$

= pick k paranthesis and use a

pick n−k paranthesis & use b

$= \binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$

$\Rightarrow (a+b)^n = \sum_{k=0}^{n} a^k b^{n-k} \binom{n}{k}$

Obs.: $\binom{n}{k} = \binom{n}{n-k}$

$\rightsquigarrow$ # of ways to make a team of k players out of n

= # of ways to pick $n-k$ people & not use them in team

Obs. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k+1}$

# of ways to make a team of $k+1$ player out of $n+1$

= either include last player OR not.

Obs. Suppose $\Omega = n$. Number of ways to partition Ω into k sets $A_1, \dots, A_k$ s.t. $|A_i| = r_i$ equals:

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3} \cdots \binom{n-r_1-r_2-\dots-r_{k-1}}{r_k}$$

Notice that since $A_1, \dots, A_k$ is a partition, i.e,

$$\Omega = A_1 \cup \dots \cup A_k \text{ and } A_i \cap A_j = \emptyset \text{ for } j \neq i$$

we have $r_1 + r_2 + \dots + r_k = n$.

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e.3. Consider the word disciplinary:

- How many words of length 12 are there?

  Starting from constraints: $\underbrace{\binom{12}{3}}_{\text{location of "i"s}} \times \underbrace{9!}_{\text{order the rest}}$

- How many are possible s.t. all i's are next to each other?

  Starting from constraints: $\underbrace{1}_{\text{construct block}} \times \underbrace{9!}_{\text{order them}}$, Starting from constraints: $\underset{\substack{\text{construct block} \\ \boxed{iii}\text{, call it } X}}{\underline{\quad 1 \quad}} \times \underset{\substack{\text{order them} \\ \{d,s,c,p,l,n,r,y,X\}}}{\underline{\quad 9! \quad}}$

. How many different words of length 12 are there so that there is exactly one letter between d and s?

If the letter between d and s is "i":

$$\underset{\substack{\text{construct the block} \\ \boxed{dis} \text{ or } \boxed{sid} \\ \text{call the block } X}}{\underline{\quad 2 \quad}} \times \underset{\substack{\text{construt a word} \\ \text{with} \\ \{i,i,c,p,l,n,r,y,X\}}}{\underline{\binom{9}{2} \cdot 7!}}$$

If the letter between d and s is not "i":

$$\underbrace{\underset{\substack{\text{pick the letter} \\ \text{between d \& s}}}{\underline{\quad 7 \quad}} \times \underset{\substack{\text{order d} \\ \text{\& s}}}{\underline{\quad 2 \quad}}}_{\substack{\text{construct block} \\ \text{call it } X}} \times \underset{\substack{\text{construct a word} \\ \text{with other letter} \\ \text{and } X}}{\underline{\binom{9}{3} \times 6!}}$$

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② Probability with equal outcomes:

If all elements of Ω are equiprobable, then $\forall A \subset \Omega$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\#\text{ elements of } A}{\#\text{ elements of } \Omega}$$

e.g. We have n red balls and m blue balls. Pick 2 balls at random.

What is $\mathbb{P}(\{\text{balls have same color}\})$?

Let us assign nubers to balls:

$$\text{red balls}: r_1, r_2, \dots, r_n$$

$$\text{blue balls}: b_1, b_2, \dots, b_m$$

$\Omega = \{\{a,b\} : a,b \in \{r_1,\dots,r_n,b_1,\dots,b_m\}\}$ = set of all possible outcomes.

$A = \{\text{both balls are red}\}$, $|A| = \binom{n}{2}$

$B = \{\text{both balls are blue}\}$, $|B| = \binom{m}{2}$

\# of ways to pick 2 balls such that they have the same color

$$= |A \cup B| = |A| + |B|$$

$$P(\{\text{balls have same color}\}) = \frac{\#\text{ of ways to pick 2 balls of same color}}{\text{total }\#\text{ of ways to pick 2 balls}}$$

$$= \frac{\binom{m}{2} + \binom{n}{2}}{\binom{m+n}{2}}$$

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③ Karnaugh map with three sets:

Recall that for two sets A, B:

	A	A^c
B	AB	A^cB
B^c	AB^c	A^cB^c

Now for three sets A & B & C:

- there are 8 different combinations of form

$$(A \text{ or } A^c) \cap (B \text{ or } B^c) \cap (C \text{ or } C^c)$$

$\Rightarrow$ there should be 8 small squars.

- each small square corresponds to one combination.

A	A	A^c	A^c	
ABC^c	ABC	A^cBC	A^cBC^c	B
AB^cC^c	ACB^c	A^cB^cC	$A^cB^cC^c$	B^c
C^c	C	C	C^c	