Lecture 29 - 10/31

Monday, October 31, 2022 2:37 AM

Review

① joint cdf.

Def. For two random variables X and Y that are defined over the same probability space:

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = P\{\omega \in \Omega : X(\omega) \leq u, Y(\omega) \leq v\}$$

② joint pmf

If X and Y are discrete-type,

$$P_{X,Y}(u,v) = P(X=u, Y=v) = P(\{\omega \in \Omega : X(\omega) \in u, Y(\omega) \in v\})$$

there exists $\{u_1, u_2, \dots\}$ and $\{v_1, v_2, \dots\}$ such that $P_{X,Y}(u,v) = 0$ if $u \in \{u_1, u_2, \dots\}$ or $v \in \{v_1, v_2, \dots\}$.

. Conditional pmf.

$$P_{Y|X}(v|u) = P(Y=v \mid X=u) = \frac{P(X=u, Y=v)}{P(X=u)} = \frac{P_{X,Y}(u,v)}{P_X(u)}$$

③ joint pdf

We say random variables X and Y are jointly continuous if

$$F_{X,Y}(u_0,v_0) = \int_{-\infty}^{u_0}\int_{-\infty}^{v_0} f_{X,Y}(u,v)\,dv\,du$$

the function $f_{X,Y}$ is called the joint pdf.

. For any region $A \subset \mathbb{R}^2$, we have

$$P((X,Y) \in A) = \iint_A f_{X,Y}(u,v)\,du\,dv$$

. Given a function $g : \mathbb{R}^2 \to \mathbb{R}$, we can use LOTUS to calcula-e $E[g(X,Y)]$:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\, f_{X,Y}(u,v)\,du\,dv$$

. Def. conditional pdf of Y given X, denoted by $f_{Y|X}$ is defined as

$$f_{Y|X}(v_0|u_0) = \begin{cases} \dfrac{f_{X,Y}(u_0,v_0)}{f_X(u_0)} & \text{if } f_X(u_0) > 0 \\ \text{undefined} & \text{o.w.} \end{cases} \quad . \text{ for any } (u_0, v_0) \in \mathbb{R}^2$$

$$f_{Y|X}(v_0|u_0) = \begin{cases} \quad \text{''} \\ \text{undefined} \quad \text{o.w.} \end{cases}, \text{ for any } (u_0, v_0) \in \mathbb{R}$$

$$P(Y \in A | X = u_0) = \int_A f_{Y|X}(v|u_0)\,dv \rightsquigarrow \text{conditional probability of } Y \text{ given } X = u$$

$$E[Y|X=u_0] = \int_{-\infty}^{+\infty} v \cdot f_{Y|X}(v|u_0)\,dv \rightsquigarrow \text{conditional expectation of } Y \text{ given } X = u$$

Today: ① independence of random variables

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① independence of random variables

Recall that events $A, B \subset \Omega$ are independent if $P(AB) = P(A)P(B)$.

Def: Random variables X and Y are defined to be independent if any pairs of events $\{X \in A\}$ & $\{Y \in B\}$ are independent.

$$P(X \in A, Y \in B) = P(X \in A)\,P(Y \in B)$$

Let $A = (-\infty, u_0)$ and $B = (-\infty, v_0)$, then we have $F_{X,Y}(u_0, v_0) = F_X(u_0) F_Y(v_0)$.

Proposition:

- random variables X & Y are independent if & only if for any $u_0, v_0 \in \mathbb{R}$:

$$F_{X,Y}(u_0, v_0) = F_X(u_0) F_Y(v_0)$$

- discrete-type random variables X & Y are independent if & only if

$$p_{X,Y}(u,v) = p_X(u)\,p_Y(v)$$

- jointly continuous-type random variables X & Y are independent if and only if

$$f_{X,Y}(u,v) = f_X(u) f_Y(v)$$

proof: Notice that $F_{X,Y}$ uniquely defines joint probability of X and Y, $P(X \in A, Y \in B)$ for any A and B.

- Determining from a joint pdf whether independence holds

Approach 1: given $f_{X,Y}(u,v)$, calculate $f_X(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,dv$. Derive the ratio $\frac{f_{X,Y}(u,v)}{f_X(u)}$. X & Y are independent if and only if the ratio does not depend on u for all $u, v \in \mathbb{R}$ (as long as $f_X(u) > 0$)

Proposition: X & Y are independent if & only if for any $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all $v \in \mathbb{R}$.

Notice that if $f_X(u) = 0$ then $f_{X,Y}(u,v) = 0$. If $f_X(u) \neq 0$ then $f_{Y|X}(v|u) = f_Y(v)$ implies $\frac{f_{X,Y}(u,v)}{\cdot} = f_Y(v)$.

Proposition: X & Y are independent if & only if for any $u \in \mathbb{R}$, either $f_X(u)=0$ or $f_{Y|X}(v|u)=f_Y(v)$ for all $v \in \mathbb{R}$.

Notice that if $f_X(u)=0$ then $f_{X,Y}(u,v)=0$. If $f_X(u) \neq 0$ then $f_{Y|X}(v|u)=f_Y(v)$ implies $\frac{f_{X,Y}(u,v)}{f_X(u)} = f_Y(v)$.

Approach 2: support of f_{XY} & swap test.

Suppose that A and B are each finite union of intervals, e.g., $A=[-1,1] \cup (3,5)$ and $B=\{4\} \cup (3,6]$.

Define $|A|$ to be the sum of the lengths of the intervals making up A, e.g., $|A|=2+2$ and $|B|=0+3$.

Define $A \times B = \{(u,v) : u \in A, v \in B\}$. $A \times B$ is called the product set. The area of product set is denoted by $|A \times B|$ and is equal to $|A| \times |B|$

Def. We say $S \subset \mathbb{R}^2$ has swap property if for any $(a,b) \in S$ and $(c,d) \in S$ we have $(a,d) \in S$ and $(b,c) \in S$

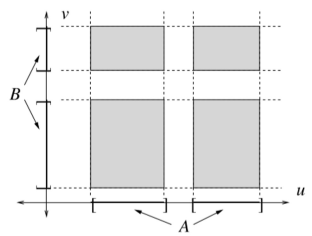
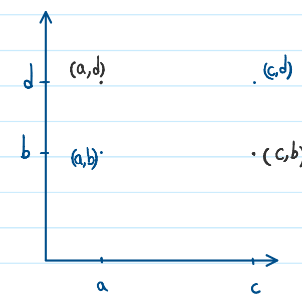


Figure 4.12: The product set $A \times B$ for sets A and B.



Notice that swap holds for product sets.

Proposition: Let $S \subset \mathbb{R}^2$. Then S is a product set if & only if it has swap property.

Suppose that $f_{XY}(u,v) = f_X(u) f_Y(v)$ for all $u,v \in \mathbb{R}$. Notice that $f_{X,Y}(u,v) > 0$ if and only if $f_X(u) > 0$ and $f_Y(v) > 0$. Hence, if $A = \{u \in \mathbb{R} : f_X(u) > 0\}$ and $B = \{v \in \mathbb{R} : f_Y(v) > 0\}$ then we have

$$\text{support of } f_{X,Y} = \{(u,v) \in \mathbb{R}^2 : f_{X,Y}(u,v) > 0\} = A \times B$$

which is a product set!

Proposition: If X & Y are independent jointly continuous, then support of $f_{X,Y}$ is a product set.

proof. It also follows by swap test. Since they are independent, we have

$$\begin{array}{c} f_{X,Y}(a,b) > 0 \\ f_{X,Y}(c,d) > 0 \end{array} \iff \begin{array}{c} f_X(a) f_Y(b) > 0 \\ f_X(c) f_Y(d) > 0 \end{array} \iff \begin{array}{c} f_X(a) f_Y(d) > 0 \\ f_X(c) f_Y(b) > 0 \end{array} \iff \begin{array}{c} f_{X,Y}(a,d) > 0 \\ f_{X,Y}(c,b) > 0 \end{array}$$

Corollary: Suppose that (X,Y) is uniformly distributed over a set S. Then X & Y are independent if & only if S is product space.

proof. If they are independent then S has to be product set. If S is product set $A \times B$ then $|S| = |A \times B| = |A||B|$ & we have

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{|A||B|} & u \in A,\ v \in B \\ 0 & \text{O.W.} \end{cases}$$

now one can integrate and derive the marginals.

4.7. [Recognizing independence]

Decide whether X and Y are independent for each of the following three joint pdfs. If they are independent, identify the marginal pdfs f_X and f_Y. If they are not, give a reason why.

(a) $f_{X,Y}(u,v) = \begin{cases} \frac{4}{\pi} e^{-(u^2+v^2)} & u, v \geq 0 \\ 0 & \text{else.} \end{cases}$

(b) $f_{X,Y}(u,v) = \begin{cases} -\frac{ln(u)v^2}{21} & 0 \leq u \leq 1, 1 \leq v \leq 4 \\ 0 & \text{else.} \end{cases}$

(c) $f_{X,Y}(u,v) = \begin{cases} \frac{(96)u^2v^2}{\pi} & u^2 + v^2 \leq 1 \\ 0 & \text{else.} \end{cases}$