Review

\( \mathcal{C}_{\text{joint cdf}} \)

- For two random variables \( X \) and \( Y \) that are defined over the same probability space:

\[
F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = P \left( \{ \omega \in \Omega : X(\omega) \leq u, Y(\omega) \leq v \} \right)
\]

\( \mathcal{C}_{\text{joint pdf}} \)

- If \( X \) and \( Y \) are discrete-type:

\[
P_{X,Y}(u,v) = P(X = u, Y = v) = P(\{ \omega \in \Omega : X(\omega) = u, Y(\omega) = v \})
\]

there exists \( \{u_1, u_2, \ldots\} \) and \( \{v_1, v_2, \ldots\} \) such that \( P_{X,Y}(u,v) = 0 \) if \( u \not\in \{u_1, u_2, \ldots\} \) or \( v \not\in \{v_1, v_2, \ldots\} \).

- Conditional pdf,

\[
P_{Y|X}(v|u) = \frac{P(Y = v, X = u)}{P(X = u)} = \frac{P_{X,Y}(u,v)}{p_X(u)}
\]

\( \mathcal{C}_{\text{joint pdf}} \)

- We say random variables \( X \) and \( Y \) are jointly continuous if

\[
F_{X,Y}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(w_1, w_2) \, dw_1 \, dw_2
\]

the function \( f_{X,Y} \) is called the joint pdf.

- For any region \( A \subset \mathbb{R}^2 \), we have

\[
P((X,Y) \in A) = \iint_A f_{X,Y}(w_1, w_2) \, dw_1 \, dw_2
\]

- Given a function \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \), we can use LOTUS to calculate \( \mathbb{E}[g(X,Y)] \):

\[
\mathbb{E}[g(X,Y)] = \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} g(w_1, w_2) f_{X,Y}(w_1, w_2) \, dw_1 \, dw_2
\]

- Def. conditional pdf of \( Y \) given \( X \), denoted by \( f_{Y|X} \), is defined as

\[
f_{Y|X}(w|u) = \begin{cases} \frac{f_{X,Y}(u,w)}{f_X(u)} & \text{if } f_X(u) > 0 \\ \text{undefined} & \text{for any } (u,w) \in \mathbb{R}^2 \end{cases}
\]
$$f_{Y|X}(u|w_u) = \begin{cases} \frac{1}{u} & \text{for any } (u, w_u) \in \mathbb{R} \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$P(Y \in A | X = w_u) = \int_A f_{Y|X}(u|w_u) \, du \quad \text{conditional probability of } Y \text{ given } X = w_u$$

$$E[Y | X = w_u] = \int_{-\infty}^{\infty} u \cdot f_{Y|X}(u|w_u) \, du \quad \text{conditional expectation of } Y \text{ given } X = w_u$$

Today: independence of random variables.

Recall that events $A, B, C, D$ are independent if $P(AB) = P(A)P(B)$.

**Definition.** Random variables $X$ and $Y$ are defined to be independent if any pairs of events $\{X \in A\}$ and $\{Y \in B\}$ are independent.

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

Let $A = (-\infty, w)$ and $B = (-\infty, u)$, then we have $F_{X,Y}(w, u) = F_X(w)F_Y(u)$.

**Proposition.**

- Random variables $X$ and $Y$ are independent if and only if for any $u, w \in \mathbb{R}$,

  $$F_{X,Y}(u, w) = F_X(u)F_Y(w)$$

- Discrete-type random variables $X$ and $Y$ are independent if and only if

  $$P_{X,Y}(u, w) = P_X(u)P_Y(w)$$

- Jointly continuous-type random variables $X$ and $Y$ are independent if and only if

  $$f_{X,Y}(u, w) = f_X(u)f_Y(w)$$

**Proof.** Notice that $F_{X,Y}$ uniquely defines joint probability of $X$ and $Y$, $P(X \in A, Y \in B)$ for any $A$ and $B$.

Determining from a joint pdf whether independence holds.

**Approach 1.** Given $f_{X,Y}(w, u)$, calculate $f_X(w) = \int_{-\infty}^{\infty} f_{X,Y}(w, u) \, du$. Derive the ratio $\frac{f_{X,Y}(w, u)}{f_X(w)}$. $X$ and $Y$ are independent if and only if the ratio does not depend on $u$ for all $u, w \in \mathbb{R}$ (as long as $f_X(w) > 0$).

**Proposition.** $X$ and $Y$ are independent if and only if for any $w \in \mathbb{R}$, either $f_X(w) = 0$ or $f_{X,Y}(u, w) = f_Y(u)$ for all $u \in \mathbb{R}$.

Notice that if $f_X(w) > 0$ then $f_{X,Y}(u, w) = 0$. If $f_X(w) > 0$ then $f_{X,Y}(u, v) = f_Y(u)$ implies $\frac{f_{X,Y}(u, v)}{f_Y(u)} = f_X(w)$. 
Proposition. Let \( S \subset \mathbb{R}^2 \). Then \( S \) is a product set if and only if it has swap property.

Suppose that \( f_{XY}(w,v) = f_X(w) f_Y(v) \) for all \( w,v \in \mathbb{R} \). Notice that \( f_{XY}(w,v) > 0 \) if and only if \( f_X(w) > 0 \) and \( f_Y(v) > 0 \). Hence, if \( A = \{ w \in \mathbb{R} : f_X(w) > 0 \} \) and \( B = \{ v \in \mathbb{R} : f_Y(v) > 0 \} \), then we have

\[
\text{support of } f_{XY} = \{ (w,v) \in \mathbb{R}^2 : f_{XY}(w,v) > 0 \} = A \times B
\]

which is a product set!

Proposition. If \( X \) & \( Y \) are independent jointly continuous, then support of \( f_{XY} \) is a product set.

proof. It also follows by swap test. Since they are independent, we have

\[
\begin{align*}
\text{support } & f_{XY} (a,b) > 0 \quad f_{XY} (a) f_Y(b) > 0 \quad f_X(a) f_Y(d) > 0 \quad f_{XY} (ad) > 0 \\
& f_{XY} (c,d) > 0 \quad f_X(a) f_Y(d) > 0 \quad f_X(c) f_Y(b) > 0 \quad f_{XY} (cb) > 0
\end{align*}
\]

Corollary. Suppose that \((X,Y)\) is uniformly distributed over a set \( S \). Then \( X \) & \( Y \) are independent if and only if \( S \) is product space.

proof. If they are independent then \( S \) has to be product set. If \( S \) is product set \( A \times B \) then \( |S| = |A| \times |B| = |A||B| \) & we have
\[
 f_{X,Y}(u,v) = \begin{cases} 
 \frac{1}{|A\cap B|}, & u \in A, v \in B \\
 0, & \text{otherwise} 
\end{cases}
\]

now one can integrate and derive the marginals.

4.7. [Recognizing independence]

Decide whether \( X \) and \( Y \) are independent for each of the following three joint pdfs. If they are independent, identify the marginal pdfs \( f_X \) and \( f_Y \). If they are not, give a reason why.

(a) \( f_{X,Y}(u,v) = \begin{cases} 
 \frac{4}{\pi} e^{-(u^2+v^2)}, & u, v \geq 0 \\
 0, & \text{else.} 
\end{cases} \)

(b) \( f_{X,Y}(u,v) = \begin{cases} 
 -\frac{\ln(u)v^2}{21}, & 0 \leq u \leq 1, 1 \leq v \leq 4 \\
 0, & \text{else.} 
\end{cases} \)

(c) \( f_{X,Y}(u,v) = \begin{cases} 
 \frac{(96)u^2v^2}{\pi}, & u^2 + v^2 \leq 1 \\
 0, & \text{else.} 
\end{cases} \)