Lecture 28 - 10/28

Wednesday, October 26, 2022 1:18 PM

Review.

① joint pmf

. If X and Y are discrete type, their joint pmf is defined as

$$P_{X,Y}(u,v) = P(X=u\ Y=v) = P(\{\omega\in\Omega: X(\omega)=u, Y(\omega)=v\})$$

there exists $\{u_1,u_2,\dots\}$ and $\{v_1,v_2,\dots\}$ such that $P_{X,Y}(u,v)=0$ if $u\notin\{u_1,u_2,\dots\}$ or $v\notin\{v_1,v_2,\dots\}$.

. Conditional probability

$$P_{Y|X}(v|u) = P(Y=v|X=u) = \frac{P(X=u, Y=v)}{P(X=u)} = \frac{P_{X,Y}(u,v)}{P_X(u)}$$

. Relation between marginal pmf and joint pmf

$$P_X(u) = \sum_i P_{X,Y}(u,v_i), \quad P_Y(v) = \sum_i P_{X,Y}(u_i,v)$$

② joint pdf

We say random variables X and Y are jointly continuous if

$$F_{X,Y}(u_0,v_0) = \int_{-\infty}^{u_0}\int_{-\infty}^{v_0} f_{X,Y}(u,v)\,dv\,du$$

the function $f_{X,Y}$ is called the joint pdf.

. For any region $A\subset\mathbb{R}^2$, we have

$$P((X,Y)\in A) = \iint_A f_{X,Y}(u,v)\,du\,dv$$

Today: joint pdf

Proposition. A function f is joint pdf if and only if:

JPD1. f is non-negative, i.e., $f(u,v)\ge 0$ for all $(u,v)\in\mathbb{R}^2$

JPD 2: $\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,du\,dv = 1$

Proof. If f satisfies above then $F(u_0,v_0) = \int_{-\infty}^{v_0}\int_{-\infty}^{u_0} f(u,v)\,du\,dv$ satisfies JF1 – JF6. If f is joint pdf, then JPD1 is holds since $P((X,Y)\in R) = \iint_R f(u,v)\,du\,dv \ge 0$ for all $R\subset\mathbb{R}^2$ (measurable)

Def: Support of $f_{X,Y}$ is the set of $(u,v)\in\mathbb{R}^2$ for which $f_{X,Y}(u,v)>0$.

. Relation between joint pdf $f_{X,Y}$ and marginal pdfs f_X and f_Y:

. $$F_X(u_0) = \lim_{v_0\to\infty} F_{X,Y}(u_0,v_0) = \lim_{v_0\to\infty}\int_{-\infty}^{v_0}\int_{-\infty}^{u_0} f_{X,Y}(u,v)\,du\,dv = \int_{-\infty}^{+\infty}\int_{-\infty}^{u_0} f_{X,Y}(u,v)\,du\,dv$$

taking derivatives with respect to u_0 yields taking derivatives with respect to v_0 yields

$$f_X(u_0) = \int_{-\infty}^{+\infty} f_{X,Y}(u_0, v)\, dv \rightsquigarrow \text{marginal pdf of } X$$

. Similarly $f_Y(v_0) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v_0)\, du \rightsquigarrow$ marginal pdf of Y

- Given a function $g: \mathbb{R}^2 \to \mathbb{R}$, we can use LOTUS to calculate $E[g(X,Y)]$:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\, f_{X,Y}(u,v)\, du\, dv$$

. Suppose that $g(X,Y) = aX + bY + c$. By LOTUS

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (au + bv + c)\, f_{X,Y}(u,v)\, du\, dv$$

$$= a\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u f_{X,Y}(u,v)\, du\, dv + b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v f_{X,Y}(u,v)\, du\, dv + c$$

$$= a\int_{-\infty}^{+\infty} u \left(\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\, dv \right) du + b\int_{-\infty}^{+\infty} v \left(\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\, du \right) dv + c$$

$$= a\int_{-\infty}^{+\infty} u f_X(u)\, du + b\int_{-\infty}^{+\infty} v f_Y(v)\, dv + c = aE[X] + bE[Y] + c$$

$\Rightarrow$ LOTUS implies linearity of expectation.

- Uniform joint pdfs. A simple class of joint pdfs are uniform joint pdfs over a surface $S \subset \mathbb{R}^2$.

$$f_{X,Y}(u,v) = \begin{cases} \dfrac{1}{\text{area of } S} & \text{if } (u,v) \in S \\ 0 & \text{if } (u,v) \notin S \end{cases} \Rightarrow P((X,Y) \in A) = \frac{\text{area of } A \cap S}{\text{area of } S}$$

- Function of (X,Y): suppose that $Z = g(X,Y)$ and we are given the joint pdf $f_{X,Y}$. Our goal is to find pdf of Z. Idea is same as before

Step 1: find support of Z, Step 2: find cdf of Z using $F_Z(z) = P(Z \leq z) = \iint_{g(X,Y) \leq z} f_{X,Y}(u,v)\, du\, dv$. Step 3: use $f_Z(u) = \frac{dF_Z}{dz}(u)$

Def: conditional pdf of Y given X, denoted by $f_{Y|X}$ is defined as

$$f_{Y|X}(v_0|u_0) = \begin{cases} \dfrac{f_{X,Y}(u_0,v_0)}{f_X(u_0)} & \text{if } f_X(u_0) > 0 \\ \text{undefined} & \text{o.w.} \end{cases}, \text{ for any } (u_0, v_0) \in \mathbb{R}^2$$

Properties of joint pdf:

- Assume $f_X(u_0) > 0$. We have,

$$\int_{-\infty}^{+\infty} f_{Y|X}(v|u_0)\, dv = \int_{-\infty}^{+\infty} f_{X,Y}(u_0, v)\, dv \quad f_X(u_0)$$

• Assume $f_X(u_0) > 0$. We have,

$$\int_{-\infty}^{+\infty} f_{Y|X}(v|u_0)\,dv = \frac{\int_{-\infty}^{+\infty} f_{X,Y}(u_0,v)\,dv}{f_X(u_0)} = \frac{f_X(u_0)}{f_X(u_0)} = 1$$

Hence, $f_{Y|X}(\cdot|u_0)$ is a valid pdf.

• We can write

$$f_{Y|X}(v_0|u_0) = \frac{f_{X,Y}(u_0,v_0)}{f_X(u_0)} = \lim_{\varepsilon \to 0} \frac{\dfrac{P\big(Y\in(v_0-\varepsilon,v_0+\varepsilon), X\in(u_0-\varepsilon,u_0+\varepsilon)\big)}{4\varepsilon^2}}{\dfrac{P\big(X\in(u_0-\varepsilon,u_0+\varepsilon)\big)}{2\varepsilon}}$$

$$= \lim_{\varepsilon \to 0} \frac{P\big(Y\in(v_0-\varepsilon,v_0+\varepsilon) \,|\, X\in(u_0-\varepsilon,u_0+\varepsilon)\big)}{2\varepsilon}$$

Hence $f_{Y|X}(u_0,v_0).2\varepsilon \approx P\big(Y\in(v_0-\varepsilon,v_0+\varepsilon) \,|\, X\in(u_0-\varepsilon,u_0+\varepsilon)\big)$, i.e., conditional pdf has probabilistic interpretation.

• $f_{X,Y}(u,v) = f_{Y|X}(v|u).f_X(u)$. This gives a version of law of total probability:

$$f_Y(v) = \int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,du = \int_{-\infty}^{+\infty} f_{Y|X}(v|u) f_X(u)\,du.$$

• Suppose that we observed $X=u_0$. It is a legitimate question to ask about statistical properties of Y.

$$P(Y\in A|X=u_0) = \int_A f_{Y|X}(v|u_0)\,dv \rightsquigarrow \text{conditional probability of } Y \text{ given } X=u$$

$$E[Y|X=u_0] = \int_{-\infty}^{+\infty} v.\,f_{Y|X}(v|u_0)\,dv \rightsquigarrow \text{conditional expectation of } Y \text{ given } X=u$$

• If we define $g(u) = E[Y|X=u]$, assuming $f_X(u) > 0$ for all u, then $g:\mathbb{R}\to\mathbb{R}$ is well-defined function. Hence, $g(X)$ is a well-defined random variable.

$$g(X) = E[Y|X] \rightsquigarrow \text{conditional expectation of } Y \text{ given } X.$$

4.9. **[Working with a joint pdf I]**

Suppose two random variables X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{uv+1}{C} & \text{if } -1 \le u \le 1 \text{ and } -1 \le v \le 1, \\ 0 & \text{else.} \end{cases}$$

(a) Find the pdf of f_X (you need not find the constant C at this point).

(b) Find the constant C.

(c) For $-1 \le u_o \le 1$, find the conditional pdf $f_{Y|X}(v|u_o)$. Specify it for all real values of v.

(d) Find $E[X^m Y^n]$ for integers $m, n \ge 0$.

(e) Find $P\{X + Y \ge 1\}$.

Solution:

(a) for any $u_0 \in [-1,1]$, $f_X(u_0) = \int_{-\infty}^{+\infty} f_{X,Y}(u_0,v)\,dv = \int_{-1}^{1} \frac{u_0 v+1}{C}\,dv = \frac{2}{C}$

for any $u_0 \notin [-1,1]$, $f_X(u_0) = 0$

(b) $\int_{-\infty}^{+\infty} f_X(u)\,du = 1 \Rightarrow \int_{-1}^{+1} \frac{2}{C} = 1 \Rightarrow C = 4.$

(c) $$f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)} = \begin{cases} \frac{u_0 v+1}{2} & \text{if } v \in [-1,1] \\ 0 & \text{if } v \notin [-1,1] \end{cases}$$

(d) $$E[X^m Y^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^m v^n f_{X,Y}(u,v)\,du\,dv$$

$$= \int_{-1}^{+1}\int_{-1}^{+1} u^m v^n \cdot \frac{uv+1}{4}\,du\,dv$$

$$= \frac{1}{4}\left(\int_{-1}^{+1}\int_{-1}^{+1} u^{m+1} v^{n+1}\,du\,dv + \int_{-1}^{+1}\int_{-1}^{+1} u^m v^n\,du\,dv\right) = \begin{cases} \frac{1}{(m+1)(n+1)} & \text{if } m \text{ and } n \text{ are both odd} \\ \frac{1}{mn} & \text{if } m \text{ and } n \text{ are both even} \\ 0 & \text{o.w.} \end{cases}$$

(e) $$P(X+Y \ge 1) = \int_0^{+1}\int_{1-v}^{1} \frac{uv+1}{4}\,du\,dv = \int_{-1}^{+1} \frac{1}{4} v\left(2v - \frac{v^2}{2}\right)dv = \frac{1}{3}$$