Lecture 27 - 10/26

Tuesday, October 25, 2022 5:57 PM

Review:

① joint cdf.

Def. For two random variables X and Y that are defined over the same probability space:

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = P\{\omega \in \Omega : X(\omega) \leq u, Y(\omega) \leq v\}$$

Properties:

- $P(a < X \leq b,\ c < Y \leq d) = F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c)$

- $\lim\limits_{u \to +\infty} F_{X,Y}(u,v) = F_Y(v)$, $\lim\limits_{v \to +\infty} F_{X,Y}(u,v) = F_X(u)$

Proposition: A function F is a joint cdf if and only if:

JF.1: $0 \leq F(u,v) \leq 1$ for any $(u,v) \in \mathbb{R}^2$

JF.2: $F(u,v)$ is nondecreasing in u and nondecreasing in v.

JF.3: $F(u,v)$ is right-continuous in u and right-continuous in v.

JF.4: $\lim\limits_{u \to -\infty} F(u,v) = 0$ and $\lim\limits_{v \to -\infty} F(u,v) = 0$

JF.5: $\lim\limits_{u \to \infty} \lim\limits_{v \to \infty} F(u,v) = 1$

JF.6. For any $a<b$ and $c<d$, $F(b,d) - F(a,d) - F(b,c) + F(a,c) \geq 0$

② joint pmf

If X and Y are discrete-type,

$$P_{X,Y}(u,v) = P(X=u, Y=v) = P(\{\omega \in \Omega : X(\omega) \in u, Y(\omega) \in v\})$$

Properties:

$$P_X(u) = \sum_i P_{X,Y}(u, v_i), \quad \text{where } P(Y \in \{v_1, v_2, \dots\}) = 1.$$

Today:

① joint pmf

② joint pdf

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① joint pmf (continued)

Def. Suppose that X and Y are discrete-type random variables defined over a same probability space $(\Omega, \mathcal{F}, P)$.

Def. Suppose that X and Y are discrete-type random variables defined over a same probability space $(\Omega, \mathcal{F}, P)$. The conditional pmf of Y given X is denoted by $p_{Y|X}$ and defined as:

$$p_{Y|X}(v|u) = P(Y=v \mid X=u) = \frac{P(X=u, Y=v)}{P(X=u)} = \frac{p_{X,Y}(u,v)}{p_X(u)}$$

Proposition: A function p is joint pmf if and only if:

JPM1. p is non-negative

JPM2: There are finite or countably infinite sets $\{u_1, u_2, \dots\}$ and $\{v_1, v_2, \dots\}$ such that $p(u,v)=0$ if $u \notin \{u_1, u_2, \dots\}$ or $v \notin \{v_1, v_2, \dots\}$

JPM3. $\sum_i \sum_j p(u_i, v_j) = 1$

Proof. If p satisfies above then $F(u,v) = \sum_{i: u_i \le u} \sum_{j: v_j \le v} p(u_i, v_j)$ satisfies JF1 – JF6. If p is joint pmf, then above are trivial.

Important remark:

- Recall that two random variables X and Y are independent if and only if $P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$ for any A, B.
  Recall that if X and Y are discrete-type, then X and Y are independent if

$$p_{X,Y}(u,v) = P(X=u, Y=v) = P(X=u) P(Y=v) = p_X(u) p_Y(v)$$

- Recall that given $E_1, E_2, \dots$ for which $E_1 \cup E_2 \cup \dots = \Omega$, $E_i \cap E_j = \emptyset$ for all $i \neq j$, and $P(E_i) \neq 0$ for all i, we have the law of total probability:

$$P(A) = \sum_{i=1}^{\infty} P(A \cap E_i) = \sum_{i=1}^{\infty} P(E_i) P(A \mid E_i)$$

Simillarly, we can write a version of law of total probability for conditional pmf.

$$p_Y(v) = \sum_i p_{X,Y}(u_i, v) = \sum_i p_X(u_i) \cdot p_{Y|X}(v|u_i)$$

4.1. **[A joint pmf]**

The joint pmf $p_{X,Y}(u,v)$ of X and Y is shown in the table below.

	u=0	u=1	u=2	u=3
v=4	0	0.1	0.1	0.2
v=5	0.2	0	0	0
v=6	0	0.2	0.1	0.1

(a) Find the marginal pmfs $p_X(u)$ and $p_Y(v)$.

(b) Let $Z = X + Y$. Find p_Z, the pmf of Z.

(c) Are X and Y independent random variables? Justify your answer.

(d) Find $p_{Y|X}(v|3)$ for all v and find $E[Y|X = 3]$.

Solution:

(a) P_X :

	u=0	u=1	u=2	u=3
$P_X(u)$	0.2	0.3	0.2	0.3

, P_Y :

	v=4	v=5	v=6
$P_Y(v)$	0.4	0.2	0.4

(b) $Z = X + Y$. Notice that Z can take values between 4 and 9

	w=4	w=5	w=6	w=7	w=8	w=9
$P_Z(w)$	0	0.3	0.1	0.4	0.1	0.1

(c) No, since $P_{X,Y}(0,4) = 0$, however, $P_X(0)\, P_Y(4) = 0.2 \times 0.4$

(d)

	v=4	v=5	v=6
$P_{Y\|X}(v\|3)$	$\frac{2}{3}$	0	$\frac{1}{3}$

$E[Y|v=3] = 4 \cdot P_{Y|X}(4|3) + 5\, P_{Y|X}(5|3) + 6\, P_{Y|X}(6|3) = \frac{14}{3}$

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② joint pdf

Def: We say random variables X and Y (that are defined over the same probability space) are jointly continuous-type if there exists a function $f_{X,Y}$, called the joint probability density function (pdf), such that

$$F_{X,Y}(u_0, v_0) = \int_{-\infty}^{u_0} \int_{-\infty}^{v_0} f_{X,Y}(u,v)\, dv\, du.$$

- Recall that for any $A \subset \mathbb{R}$ (A measurable), we have $P(X \in A) = \int_A f_X(u)\, du$

Simillarly, for any $R \subset \mathbb{R}^2$ (R measurable), we have $P((X,Y)\in R) = \iint_R f_{X,Y}(u,v)\,du\,dv$. In particular,

$$P(a < X \leq b,\ c < Y \leq d) = F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c)$$
$$= \int_c^d \int_a^b f_{X,Y}(u,v)\,du\,dv$$

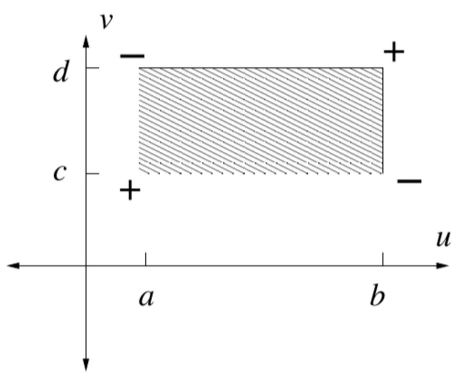

Figure 4.2: $P\{(X,Y) \in \text{shaded region}\}$ is equal to $F_{X,Y}$ evaluated at the corners with signs shown.

which also equals $\int_c^d \int_a^b f_{X,Y}(u,v)\,du\,dv$, given X and Y are jointly continuous.

Suppose that $\varepsilon > 0$ is very small and assume that $f_{X,Y}(u,v)$ is continuous in both X and Y. Using above equality, we have

$$P\Big(X \in (u_0-\varepsilon, u_0+\varepsilon),\ Y \in (v_0-\varepsilon, v_0+\varepsilon)\Big) \approx f_{X,Y}(u_0,v_0)\,(2\varepsilon)^2$$

In particular,

$$f_{X,Y}(u_0,v_0) = \lim_{\varepsilon \to 0} \frac{P\Big(X \in (u_0-\varepsilon, u_0+\varepsilon),\ Y \in (v_0-\varepsilon, v_0+\varepsilon)\Big)}{(2\varepsilon)^2}$$

This gives a probabilistic interpretation of joint pdf $f_{X,Y}$.

Important remark:

Since X and Y are continuous type, $P(X=u) = P(Y=v) = 0$, for any $u, v \in \mathbb{R}$. In particular,

$$P(a \leq X \leq b,\ c \leq Y \leq d) = P(a < X < b,\ c < Y < d) \quad \text{for any } a<b \text{ and } c<d$$