Review.

**Joint cdf.**

**Def.** For two random variables $X$ and $Y$ that are defined over the same probability space,

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = P\{w \in \Omega : X(w) \leq u, Y(w) \leq v\}$$

**Properties.**

- $P(a < X \leq b, c < Y \leq d) = F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c)$
- \[ \lim_{u \to \infty} F_{X,Y}(u,v) = F_Y(v), \quad \lim_{v \to \infty} F_{X,Y}(u,v) = F_X(u) \]

**Proposition.** A function $F$ is a joint cdf if and only if:

**JF.1.** $0 \leq F(u,v) \leq 1$ for any $(u,v) \in \mathbb{R}$

**JF.2.** $F(u,v)$ is nondecreasing in $u$ and nondecreasing in $v$.

**JF.3.** $F(u,v)$ is right-continuous in $u$ and right-continuous in $v$.

**JF.4.** \[ \lim_{u \to \infty} F(u,v) = 0 \text{ and } \lim_{v \to \infty} F(u,v) = 0 \]

**JF.5.** \[ \lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = 1 \]

**JF.6.** For any $a < b$ and $c < d$, $F(b,d) - F(a,d) - F(b,c) + F(a,c) \geq 0$

@ joint pmf

If $X$ and $Y$ are discrete-type,

$$P_{X,Y}(u,v) = P(X = u, Y = v) = P\{w \in \Omega : X(w) = u, Y(w) = v\}$$

**Properties.**

$$P_X(u) = \sum_i P_{X,Y}(u,u_i), \quad \text{where} \ P(Y \in \{v_1, v_2, \ldots \}) = 1.$$

Today.

@ joint pmf

@ joint pdf

@ joint pmf (continued)

**Def.** Suppose that $X$ and $Y$ are discrete-type random variables defined over a same probability space $(\Omega, F, P)$.
Def. Suppose that $X$ and $Y$ are discrete-type random variables defined over a same probability space $(\Omega, F, P)$. The conditional pmf of $Y$ given $X$ is denoted by $P_{Y|X}$ and defined as:

$$P_{Y|X}(y|u) = P(Y=y \mid X=u) = \frac{P(X=u, Y=y)}{P(X=u)} = \frac{P_{XY}(u,y)}{P_X(u)}$$

Proposition: A function $p$ is joint pmf if and only if:

JPM1. $p$ is non-negative

JPM2. There are finite or countably infinite sets $\{u_1, u_2, \ldots\}$ and $\{v_1, v_2, \ldots\}$ such that $p(u,v) = 0$ if $u \notin \{u_1, u_2, \ldots\}$ or $v \notin \{v_1, v_2, \ldots\}$

JPM3. $\sum_{u} \sum_{v} p(u,v) = 1$

Proof: If $p$ satisfies above then $F(u,v) = \sum_{u} \sum_{v} p(u,v)$ satisfies JF1-JF6. If $p$ is joint pmf, then above are trivial.

Important remark:

Recall that two random variables $X$ and $Y$ are independent if and only if $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any $A, B$.

Recall that if $X$ and $Y$ are discrete-type, then $X$ and $Y$ are independent if

$$P_{XY}(u,v) = P(X=u, Y=v) = P(X=u)P(Y=v) = P_X(u)P_Y(v)$$

Recall that given $E_1, E_2, \ldots$ for which $E_1 \cup E_2 \cup \ldots = \Omega$, $E_i \cap E_j = \emptyset$ for all $i \neq j$, and $P(E_i) > 0$ for all $i$, we have the law of total probability:

$$P(A) = \sum_{i=1}^{\infty} P(A \mid E_i) = \sum_{i=1}^{\infty} P(E_i)P(A \mid E_i)$$

Similarly, we can write a version of law of total probability for conditional pmf:

$$P_{Y}(v) = \sum_{i=1}^{\infty} P_{Y|X}(v|u) = \sum_{i=1}^{\infty} P_X(u) \cdot P_{Y|X}(v|u)$$
4.1. **[A joint pmf]**

The joint pmf $p_{X,Y}(u, v)$ of $X$ and $Y$ is shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>u=0</th>
<th>u=1</th>
<th>u=2</th>
<th>u=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>v=4</td>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>v=5</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>v=6</td>
<td>0</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(a) Find the marginal pmfs $p_X(u)$ and $p_Y(v)$.
(b) Let $Z = X + Y$. Find $p_Z$, the pmf of $Z$.
(c) Are $X$ and $Y$ independent random variables? Justify your answer.
(d) Find $p_{Y|X}(v|3)$ for all $v$ and find $E[Y|X = 3]$.

**Solution:**

(a) $p_X$:

<table>
<thead>
<tr>
<th></th>
<th>u=0</th>
<th>u=1</th>
<th>u=2</th>
<th>u=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_X(u)$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
</tr>
</tbody>
</table>

(b) $Z = X + Y$. Notice that $Z$ can take values between 4 and 9.

<table>
<thead>
<tr>
<th></th>
<th>w=4</th>
<th>w=5</th>
<th>w=6</th>
<th>w=7</th>
<th>w=8</th>
<th>w=9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_Z(w)$</td>
<td>0</td>
<td>0.3</td>
<td>0.1</td>
<td>0.4</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(c) No, since $p_{X,Y}(0, 4) = 0$, however, $p_X(0)p_Y(4) = 0.2 	imes 0.4$

(d)

<table>
<thead>
<tr>
<th></th>
<th>v=4</th>
<th>v=5</th>
<th>v=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{Y</td>
<td>X}(v</td>
<td>3)$</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

$E[Y|X=3] = 4p_{Y|X}(4|3) + 5p_{Y|X}(5|3) + 6p_{Y|X}(6|3) = \frac{14}{3}$

**Joint pdf**

**Def.** We say random variables $X$ and $Y$ (that are defined over the same probability space) are jointly continuous-type if there exists a function $f_{X,Y}$, called the joint probability density function (pdf), such that

$$f_{X,Y}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(u,v) \, d\omega \, d\nu.$$

Recall that for any $A \subset \mathbb{R}$ (A measurable), we have $P(X \in A) = \int_A f_X(x) \, dx$. 

Similarly, for any \( R \subseteq \mathbb{R}^2 \) (\( R \) measurable), we have \( P((X,Y) \in R) = \int_R f_{X,Y}(u,v) \, du \, dv \). In particular,

\[
P(a \leq X \leq b, c \leq Y \leq d) = F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c)
\]

\[
= \int_c^d \int_a^b f_{X,Y}(w,v) \, dw \, dv
\]

Figure 4.2: \( P((X,Y) \in \text{shaded region}) \) is equal to \( F_{X,Y} \) evaluated at the corners with signs shown.

which also equals \( \int_c^d \int_a^b f_{X,Y}(w,v) \, dw \, dv \), given \( X \) and \( Y \) are jointly continuous.

Suppose that \( \varepsilon > 0 \) is very small and assume that \( f_{X,Y}(u,v) \) is continuous in both \( X \) and \( Y \).

Using above equality, we have

\[
P \left( X \in (u-\varepsilon, u+\varepsilon), Y \in (v-\varepsilon, v+\varepsilon) \right) \approx f_{X,Y}(u,v) (2\varepsilon)^2
\]

In particular,

\[
\int_{u_0}^{u_0} f_{X,Y}(u_0, v) = \lim_{\varepsilon \to 0} P \left( X \in (u-\varepsilon, u+\varepsilon), Y \in (v-\varepsilon, v+\varepsilon) \right) (2\varepsilon)^2
\]

This gives a probabilistic interpretation of joint pdf \( f_{X,Y} \).

Important remark.

Since \( X \) and \( Y \) are continuous type, \( P(X=u) = P(Y=v) = 0 \), for any \( u,v \in \mathbb{R} \). In particular,

\[
P(a \leq X \leq b, c \leq Y \leq d) = P(a \leq X \leq b, c \leq Y \leq d) \text{ for any } a \leq b \text{ and } c \leq d
\]