Review:

0. Distribution of a function of random variable

\[ Y = g(X), \text{ pdf if } X \text{ is given, pdf of } Y \text{ is desired} \]

Step 1: Find type of \( Y \) & its support

Step 2: If continuous type, find its cdf; if discrete type, find its pmf

Step 3: If continuous type, find its pdf by differentiating

0. Generating variable with specified distribution

Given \( U \sim \text{unif}(0,1) \) and a cdf \( F \), find function \( g \) such that the cdf of \( X = g(U) \)

is \( F \), i.e., for any \( c \in \mathbb{R} \), \( F_g(c) = F(c) \).

Answer: \( g(x) = \frac{F_x^{-1}}{F(U)} = \min \left\{ c : F(c) \geq x \right\} \).

(a) If possible calculate \( F^{-1} \)

(b) If it is hard, use reflection & then make it into a function

0. Hypothesis testing for continuous type observation

(i) The system generates continuous-type random variable \( X \)

- If the system is in state \( H_0 \), the pdf of \( X \) is given by \( f_0 \)
- If the system is in state \( H_1 \), the pdf of \( X \) is given by \( f_1 \)

(ii) We observe a realization of \( X \), i.e., we observe \( X = u \).

(iii) We guess the state of system, using our observation, based on the decision rule.

We are only interested in threshold decision rule that compare \( \Delta(u) = \frac{f_1(u)}{f_0(u)} \) with threshold \( \epsilon \).

\[ ML: \epsilon = 1, \text{ MAP: } \epsilon = \frac{f_0}{f_1} \]

Today:

0. Quick recap of cdf

0. Joint cdf

0. Joint pdf

0. Quick recap of cdf

Recall that cdf of a random variable \( X \) is defined as \( F_X(c) = P(X \leq c) \), for any \( c \in \mathbb{R} \). Also recall that a function \( F \) is a cdf if and only if it satisfies the following:

F.1. \( F \) is increasing.

F.2. \( \lim_{c \rightarrow -\infty} F(c) = 0 \), \( \lim_{c \rightarrow \infty} F(c) = 1 \)

F.3. \( F \) is right-continuous, i.e., \( \lim_{c \uparrow c_0} F(c) = F(c) \)

cdf of a random variable uniquely determines probability of any event \( P(X \in A) \) for \( A \in \mathbb{R} \). (A reasonable!)
cdf of a random variable uniquely determines probability of any event $P(X \in A)$ for $A \subseteq \mathbb{R}$ (A measurable!)

We were interested in $F_X$, since we were interested in distribution of $X$. What if we are interested in distribution of multiple random variable at the same time?

**Joint cdf**

Def. Suppose that $X$ and $Y$ are two random variables defined over the same probability space $(\mathcal{A}, \mathbb{P})$, i.e., $X: \mathbb{R} \rightarrow \mathbb{R}$ and $Y: \mathbb{R} \rightarrow \mathbb{R}$

The joint cumulative distribution function (joint cdf) is the function of two variables defined as

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) \quad \text{for any } u,v \in \mathbb{R}$$

**Example.** Suppose that $X$ denotes the temperature of your mobile while gaming, & $Y$ denotes its CPU usage. We are interested in probability of the event \{w.r.t. $X(x) \leq 50$ & $Y(y) \leq 45$\}

$$F_{X,Y}(50,45) = P(X \leq 50, Y \leq 45) = P\left(\begin{array}{l}
\text{w.r.t. } X(x) \leq 50 \\
\text{w.r.t. } Y(y) \leq 45
\end{array}\right)$$

Notice that $X$ & $Y$ are defined over the same space.

Notice that the event \{X \leq u, Y \leq v\} means $(X,Y)$ is in the shaded area below.

![Figure 4.1: Region defining $F_{X,Y}(u,v)$.](image)

Joint cdf uniquely determines the distribution of any joint event $P((X,Y) \in A)$, where $A \subseteq \mathbb{R}^2$ (A measurable!)

In particular, if $A = [a,b] \times [c,d]$ we have

$$P((X,Y) \in A) = P(a \leq X \leq b, c \leq Y \leq d) = F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(a,b) + F_{X,Y}(a,c)$$

$$= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - P(X \leq a, Y \leq d) + P(X \leq a, Y \leq c)$$

we have deducted this area twice so we have to add one back

$\Rightarrow$ sign is +
Figure 4.2: \( P\{(X, Y) \in \text{shaded region}\} \) is equal to \( F_{X,Y} \) evaluated at the corners with signs shown.

Joint cdf and marginal cdf are related to each other.

**Proposition 4.1.** Suppose that \( F_{X,Y} \) is joint cdf of \( X \) and \( Y \).

\[
\begin{align*}
\lim_{u \to \infty} F_{X_Y}(u, v) &= \lim_{u \to \infty} P(X \leq u, Y < v) = P(X \leq u) = F_X(u) \quad \text{called marginal cdf of } X \\
\lim_{u \to \infty} F_{X_Y}(u, v) &= \lim_{u \to \infty} P(X < u, Y \leq v) = P(Y \leq v) = F_Y(v) \quad \text{called marginal cdf of } Y
\end{align*}
\]

**Proof.** Let \( G_u = \{X \leq u, Y < v\} \) and define \( G_n = \{X \leq u, n-1 < Y \leq n\} \) for any \( n \geq 1 \).

\[
F_X(u) = P(X \leq u) = P(G_u) = \lim_{n \to \infty} P(G_n) = \lim_{n \to \infty} P(X \leq u, Y < v) = \lim_{u \to \infty} F_{X,Y}(u, v)
\]

Since \( F_{X,Y}(u, v) \) is non-decreasing in \( v \), \( \lim_{v \to \infty} F_{X,Y}(u, v) \) is well-defined, and hence equals \( F_X(u) \).

Similar to cdf, joint cdf also satisfies some properties. Moreover any function that satisfies these is a joint cdf.

**Proposition.** A function \( F \) is a joint cdf of some pair of random variables if and only if:

\[ JF.1: \quad 0 \leq F(u, v) \leq 1 \text{ for any } (u, v) \in \mathbb{R}^2 \]

\[ JF.2: \quad F(u, v) \text{ is non-decreasing in } u \text{ and non-decreasing in } v. \]

\[ JF.3: \quad F(u, v) \text{ is right-continuous in } u \text{ and right-continuous in } v. \]

\[ JF.4: \quad \lim_{u \to -\infty} F(u, v) = 0 \text{ and } \lim_{u \to \infty} F(u, v) = 0 \]

\[ JF.5: \quad \lim_{u \to -\infty} \lim_{v \to -\infty} F(u, v) = 0 \]

\[ JF.6: \quad \text{For any } a \leq b \text{ and } c \leq d, \quad F(b, d) - F(a, d) - F(b, c) + F(a, c) \geq 0 \]

This is due dimension! Being non-decreasing won't guarantee this.
Joint p.m.f

**Def:** Suppose that $X$ and $Y$ are two discrete-type random variables defined over the same probability space $(\Omega, \mathcal{F}, P)$. The joint probability mass function (joint p.m.f) is denoted by $p_{XY}$ and defined as:

$$p_{XY}(x,y) = P(X=x, Y=y)$$

**Remark:** Since $X$ and $Y$ are discrete type, they can only take at most countably many values.

- There exists $u_1, u_2, \ldots$ s.t. $P(X \in \{u_1, u_2, \ldots\}) = 1$
- There exists $v_1, v_2, \ldots$ s.t. $P(Y \in \{v_1, v_2, \ldots\}) = 1$

Joint p.m.f and marginal p.m.f.s are related:

- $P_X(u) = P(X=u) = P(X=u, Y \in \{v_1, v_2, \ldots\}) = \sum_i P(X=u, Y=v_i) = \sum_i p_{XY}(u,v_i)$ \hspace{1cm} marginal p.m.f of $X$
- $P_Y(v) = P(Y=v) = P(Y=v, X \in \{u_1, u_2, \ldots\}) = \sum_i P(Y=v, X=u_i) = \sum_i p_{XY}(u_i,v)$ \hspace{1cm} marginal p.m.f of $Y$

$p_X$ and $p_Y$ are called marginal p.m.f.