Lecture 22 - 10/12

Review:

① Poisson process $N = (N_t ; t>0)$ with rate $\lambda>0$

- N_t : number of arrivals till time t. $N_t \sim Pois(\lambda)$ ⇝ discrete r.v. with continuous index t
- T_r : time till r'th arrival, sum of r independent exponential r.v.s. $T_r \sim Erlang(r, \lambda)$
- U_n : inter arrival times between n'th and (n-1)'th arrival. $U_n \sim exp(\lambda)$ ⇝ continuous random variable with discrete index r

• Poisson process can be defined using $(U_n, n \geq 1)$ instead of $(N_t ; t \geq 0)$

② Scaling of random variables:

$$Y = aX + b \Rightarrow f_Y(u) = \frac{1}{a} f_X\left(\frac{u-b}{a}\right)$$

Today:

① Guassian (normal) distribution

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① Guassian (Normal) distribution

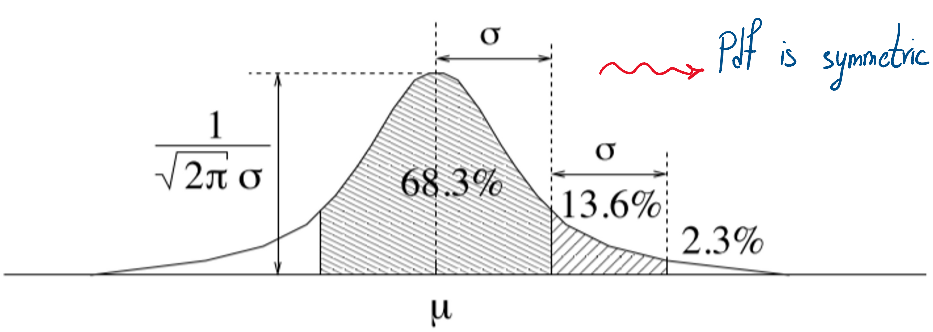
Def: We say a random variable X has Guassian distribution with parameters (μ, σ^2) if

$$f_X(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

In here parameters μ and σ^2 are special:

$$\mu = E[X] \quad \text{and} \quad \sigma^2 = Var(X)$$

We will show above equalities shortly. Notationally, we write $X \sim N(\mu, \sigma^2)$. The following figure shows the pdf:



⇝ Pdf is symmetric w.r.t. to $\mu \Rightarrow E[X] = \mu$

• $E[X] = \mu$, $f_X(\mu) = \frac{1}{\ }$

$\cdot E[X] = \mu \;,\; f_X(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}$

$\cdot P(X \in [\mu-\sigma, \mu+\sigma]) \approx \frac{68.3}{100}$

$\cdot P(X \in [\mu-2\sigma, \mu+2\sigma]) \approx \frac{68.3}{100} + 2\times\frac{13.6}{100} = \frac{95.5}{100}$

$\cdot P(X \in [\mu-3\sigma, \mu+3\sigma]) \approx \frac{99}{100}$ $\rightsquigarrow$ if σ is small the figure is very narrow

. Standardized version

Let $\tilde{X}$ denote standardized version of X, i.e., $\tilde{X} = \frac{X-\mu}{\sigma} = \frac{X}{\sigma} - \frac{\mu}{\sigma}$. We have

$$f_{\tilde{X}}(u) = \sigma f_X\left(\sigma\left(u + \frac{\mu}{\sigma}\right)\right)$$
$$= \sigma f_X(\sigma u + \mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$$

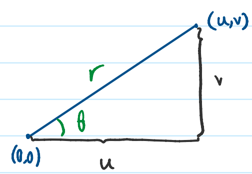
In particular, $\tilde{X} \sim \mathcal{N}(0,1)$. Traditionally this is denoted by Φ, and its complement is denoted by Q:

$$\boxed{\begin{array}{l} \cdot\; F_{\tilde{X}}(a) = P(X \le a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{u^2}{2}\right) du = \Phi(a) \\ \cdot\; F^c_{\tilde{X}}(a) = P(X > a) = \int_{a}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = 1 - \Phi(a) = Q(a) \\ \cdot\; \text{Notice that } Q(a) = \Phi(-a) \text{ because of symmetry!} \end{array}}$$

. Some proofs: Suppose that $Y \sim \mathcal{N}(0,1)$.

(i) We want to show $f_Y(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$ is pdf. The trick is to switch to polar coordinates.

$$\left(\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} du\right) = \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} du \int_{-\infty}^{+\infty} e^{-\frac{v^2}{2}} dv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{u^2+v^2}{2}} du\,dv$$
$$= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr\,d\theta$$
$$= 2\pi \cdot e^{-\frac{r^2}{2}}\Big|_0^{\infty} = 2\pi$$



$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = 1$$

(ii) Since $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$ is symmetric with respect to 0, $E[X] = 0$.

(iii) For variance,

$$V(X) = E[X^2] = \int^{+\infty} \ldots \frac{1}{\ldots} e^{-\frac{u^2}{2}} \ldots$$

(iii) For variance,

$$Var(X) = E[X^2] = \int_{-\infty}^{+\infty} u^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

$$= \frac{1}{\sqrt{2\pi}} u e^{-\frac{u^2}{2}} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du = 0 + 1 = 1 \rightsquigarrow \text{integration by part}$$

$\Rightarrow Y \sim N(0,1)$ has mean 0 and Variance 1.

(iv) using LOTUS, $X = \sigma Y + \mu$ has mean μ and variance σ^2, i.e., $X \sim N(\mu, \sigma^2)$

(v) Suppose that $X \sim N(\mu, \sigma^2)$. Let $\tilde{X}$ denote the normalized version.

$$P(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) = P\left(\tilde{X} \leq \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X > a) = P\left(\frac{X-\mu}{\sigma} > \frac{a-\mu}{\sigma}\right) = P\left(\tilde{X} > \frac{a-\mu}{\sigma}\right) = Q\left(\frac{a-\mu}{\sigma}\right) = \Phi\left(-\frac{a-\mu}{\sigma}\right)$$

Example:

3.17. **[A mixture of Gaussian distributions]**
Suppose X, Y and Z are three mutually independent random variables such that $X \sim N(2,4)$, $Y \sim N(7,9)$, and $B \sim \text{Bernoulli}(2/3)$. Let $Z = XB + Y(1-B)$. The distribution of Z is a mixture of the distributions of X and Y.

(a) Find the pdf of Z. (Hint: First find the CDF. Use the law of total probability, taking into account the two possible cases $B = 1$ and $B = 0$.)

(b) Find $P\{Z \geq 4 | B = 1\}$.

(c) Find $P\{Z = 4 | B = 1\}$.

(d) Find $P\{Z \leq 4 | B = 0\}$.

(e) Find $P\{Z \geq 4\}$.

Solution.

(a) Recall that to get the relation between the pdf of scaled random variable and original random variable, we calculated the relation between CDFs and then took the derivative to get pdf. Idea is the same here:

$$F_Z(c) = P(Z \leq c) = P(B=1)P(Z \leq c | B=1) + P(B=0)P(Z \leq c | B=0)$$

$$= P(B=1)P(X \leq c | B=1) + P(B=0)P(Y \leq c | B=0) \rightsquigarrow \text{definition of } Y$$

$$= P(B=1)P(X \leq c) + P(B=0)P(Y \leq c) \rightsquigarrow B, X, Y \text{ are independent}$$

$$= \frac{2}{3}\Phi\left(\frac{c-2}{2}\right) + \frac{1}{3}\Phi\left(\frac{c-7}{3}\right)$$

Notice that $\frac{dF_Z(c)}{dc} = f_Z(c)$ and $\frac{d\Phi(c)}{dc} = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$. Hence,

$$f_Z(c) = \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{u-2}{2}\right)^2} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{u-7}{3}\right)^2}, \text{ for any } z \in \mathbb{R}.$$

(b) $P(Z \geq 4 | B=1) = P(XB + (1-B)Y \geq 4 | B=1)$

$= P(X \geq 4 | B=1)$

$= P(X \geq 4) = P(X > 4) = Q\left(\frac{4-2}{2}\right) = Q(1) = \Phi(-1)$

(c) $P(Z = 4 | B=1) = P(X = 4 | B=1) = P(X=4) = 0$

(d) $P(Z \leqslant 4 | B=0) = P(Y \leqslant 4 | B=0) = P(Y \leqslant 4) = \Phi\left(\frac{4-7}{3}\right) = \Phi(-1)$

(e) $P(Z \leqslant 4) = P(Z \leqslant 4 | B=0) P(B=0) + P(Z \leqslant 4 | B=1) P(B=1)$

$= \frac{1}{3} \Phi(-1) + \frac{2}{3}(1 - \Phi(-1)) = \frac{2}{3} - \frac{1}{3}\Phi(-1)$