Lecture 21 - 10/10

Review:

① Poisson process

Def: A Poisson process with rate $\lambda > 0$ is a conting process $N = (N_t ; t \geq 0)$ that satisfies the followings

(N.1) It has independent incrament property: $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$, then $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent

(N2) For any $t > s$, $N_t - N_s$ has Poisson distribution with parameter $\lambda(t-s)$

(N3) $N_0 = 0$.

• Some related random variables.

(i) U_n = inter arrival time between $(n-1)$'th and n'th arrival, U_n has exponential distribution with parameter λ.

(ii) T_r = time till the r'th arrival, T_r has Erlang distribution with parameters (r, λ)

$$f_{T_r}(t) = \begin{cases} \dfrac{\lambda e^{-\lambda t} (\lambda t)^{r-1}}{(r-1)!}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$F_{T_r}(t) = 1 - \sum_{i=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}$$

Notice that $\int_{-\infty}^{+\infty} f_{T_r}(u)\, du = 1$, since it is a pdf.

• Relation between above random variables: (**)

(i) $N_t = \sum_{r=1}^{\infty} 1_{\{T_r \leq t\}}$ where $1_{\{\cdot\}}$ is indicator function.

$$1_A = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

(ii) $T_r = U_1 + U_2 + \dots + U_r$

(iii) $T_r = \min \{ t : N_t \geq r \}$

Today:

① Embedded limiting for poisson

Today:

① Equivalent definition for poisson process

② scaling rule for pdfs

① Equivalent definition for poisson process

Proposition: Let N be a random counting process and let $\lambda > 0$. The following are equivalent.

(i) N is a Poisson process with parameter λ.

(ii) The intercount (inter arrival) times $U_1, U_2, \ldots, U_n$ are mutually independent and exponentially distributed random variables.

Proof. To see this, notice that each of the above properties

(i) define a counting process. For b, we can use relations in (**) to define a counting process.

(ii) both of these process appeared as the limiting process of a time-scaled Bernoulli process.

Example:

3.11. **[Poisson process]**
A certain application in a cloud computing system is accessed on average by 15 customers per minute. Find the probability that in a one minute period, three customers access the application in the first ten seconds and two customers access the application in the last fifteen seconds. (Any number could access the system in between these two time intervals.)

Solution: Let N denote the Poisson process associated with the above system.

N_t: number of customers accessed the system during the first t seconds of operation.

- On averag 15 customer per minute: $E[N_{60}] = 15 \Rightarrow 60\lambda = 15 \Rightarrow \lambda = \frac{1}{4}$
- In one minute period 3 customers accessed in the first 10 second and 2 accessed in the last 15 seconds:

$$\{N_{10} = 3, N_{60} - N_{45} = 2\}$$

$$P(\{N_{10} = 3, N_{60} - N_{45} = 2\}) = P(N_{10} = 3) \cdot P(N_{60} - N_{45} = 2)$$

$$= \frac{e^{-2.5}(2.5)^3}{3!} \cdot \frac{e^{-3.75} \cdot (3.75)^2}{2!}$$

Now, calculate probability of n customer in the first minute given $2n$ customer appeared in the first 2 minutes:

Solution:

Now, calculate probability of n customer in the first minute given $2n$ customer appeared in the first 2 minutes:

Solution:

$$P(N_{60}=n \mid N_{120}=2n) = \frac{P(N_{60}=n, N_{120}=2n)}{P(N_{120}=2n)} = \frac{P(N_{60}=n)P(N_{120}-N_{60}=n)}{P(N_{120}=2n)}$$

$$= \frac{e^{-15}\frac{(15)^n}{n!} \times e^{-15}\frac{(15)^n}{n!}}{e^{-30}\frac{(30)^n}{(2n)!}}$$

---

② Scaling rule for pdf

Let X be a random variable with pdf f_X and let $Y = aX + b$ where $a > 0$.

$$F_Y(u) = \mathbb{P}(Y \leqslant u) = \mathbb{P}(aX+b \leqslant u) = P\left(X \leqslant \frac{u-b}{a}\right) = F_X\left(\frac{u-b}{a}\right)$$

Differentiating both sides with respect to u, we get

$$\boxed{f_Y(u) = \frac{1}{a} f_X\left(\frac{u-b}{a}\right) \rightsquigarrow \text{scaling rule for } Y = aX + b, \text{ where } a > 0}$$

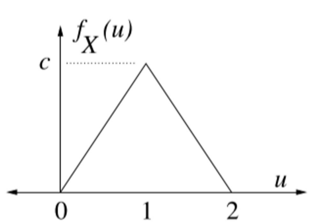
Recall that by LOTUS:

$$E[Y] = aE[X] + b$$

$$Var(Y) = a^2 Var(X)$$

Same can be obtained if we plug-in definitions of mean & variance using its pdf f_Y.

3.15. **[Standardizing a random variable with a triangular pdf]**
Suppose X has the pdf shown:



(a) Find the constant c.

(b) Let $\widetilde{X}$ denote the standardized version of X. Thus, $\widetilde{X} = \frac{X-a}{b}$ for some constants a and b so that $\widetilde{X}$ has mean zero and variance one. Carefully **sketch** the pdf of $\widetilde{X}$. Be sure to indicate both the horizontal and vertical scales of your sketch by labeling at least one nonzero point on each of the axes. (Hint: $\text{Var}(X) = \frac{1}{6}$.)

Solution: (a) Notice that $\int_0^2 f_X(u)\,du =$ area under the curve $= 1 \Rightarrow 2C \cdot \frac{1}{2} = 1 \Rightarrow C = 1$

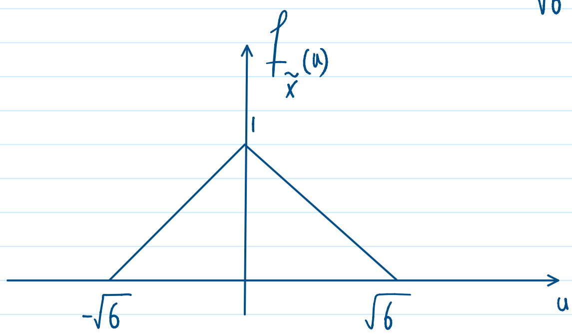
(b) Notice that

$$\tilde{X} = \frac{X - E[X]}{\sigma_X},$$

$$\sigma_X^2 = Var(X) = \frac{1}{6} \Rightarrow \sigma_X = \frac{1}{\sqrt{6}}$$

$E[X] = 1$ since it is symmetric with respect to point 1.

$$\Rightarrow \tilde{X} = \sqrt{6}\,X - \sqrt{6} \quad \text{and} \quad f_{\tilde{X}}(u) = \frac{1}{\sqrt{6}} f_X\left(\frac{u+\sqrt{6}}{\sqrt{6}}\right)$$



Important remark: Suppose that the pdf of a random variable X is symmetric with respect to point $c>0$:

$$f_X(x+c) = f_X(c-x) \quad \text{for all } x \in \mathbb{R}$$

We show that $E[X] = c$. Let $Y = X - C$. Notice that $f_Y(u) = f_X(u+c)$

$$E[Y] = \int_{-\infty}^{+\infty} u f_Y(u)\,du = \int_{-\infty}^{+\infty} u f_X(u+c)\,du = \int_0^{+\infty} u f_X(u+c)\,du + \int_{-\infty}^{0} u f_X(u+c)\,du$$

$$= \int_0^{+\infty} u f_X(u+c) - \int_0^{+\infty} (-v) f_X((-v)+c)\,dv \quad : \text{change of variable } v = -u$$

$$= \int_0^{+\infty} u f_X(u+c) + \int_0^{\infty} v f_X(c-v)\,dv = 0 \quad : \text{Since } f_X(u+c) = f_X(u-c)$$