Lecture 19 - 10/5

Wednesday, October 05, 2022 9:56 AM

Review:

① union bound. $P(A_1 \cup \cdots \cup A_n) \leqslant \sum_{i=1}^{n} P(A_i)$

② Network outage probability & distribution of capacity of flow network

③ Cummulative distribution function: $F_X(c) = \mathbb{P}(X \leqslant c) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leqslant c\})$, for any $c > 0$.

main properties: F is cdf of some random variable X if & only if:

F.1. increasing: $a < b \Rightarrow F(a) < F(b)$

F.2. $\lim_{c \to \infty} F(c) = 1$, $\lim_{c \to -\infty} F(c) = 0$

F.3. F is right continuous, i.e., $\lim_{u \to c^+} F(u) = F(c)$

other properties:

(i) $\Delta F_X(c) = F_X(c) - F_X(c-) = P(X = c)$

(ii) $F_X(c-) = \lim_{u \to c^-} F_X(u) = P(X < c)$

(iii) For any $a < b$, $F_X(b) - F_X(a) = P(a < X \leqslant b)$

④ Continuous random variable:

Def. X is continuous-type random variable if $F_X(c) = \int_{-\infty}^{c} f_X(u)\,du$

. f_X is the probability density function of X

. $f_X(u) \geq 0$ for all $u \in \mathbb{R}$, $\int_{-\infty}^{+\infty} f_X(u)\,dx = 1$

. Support of f_X is the set u for which $f_X(u) > 0$

. If f_X is continuous at u, then $F_X'(u) = f_X(u)$

. Interpretation of f_X: suppose that f_X is continuous at u,

$$P(u + \varepsilon < X < u - \varepsilon) = f_X(u) \cdot 2\varepsilon + o(\varepsilon), \text{ where } \lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$$

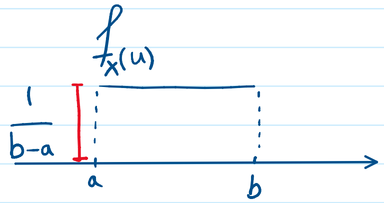
. $P(X = u) = 0$ for all continuous random variables.

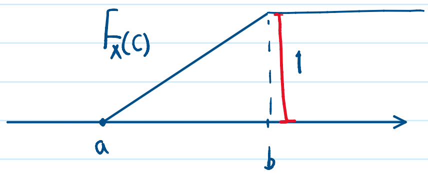
⑤ Uniform distribution

$$f_X(u) = \begin{cases} \frac{1}{b-a} & a \leqslant u \leqslant b \end{cases}, \qquad \mu_X = E[X] = \frac{b+a}{2}$$

$$f_X(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{o.w} \end{cases}, \qquad \mu_X = E[X] = \frac{b+a}{2}$$

$$var(X) = \frac{(b-a)^2}{12}$$





Today:

① exponential distribution

② relation between geometric distribution & exponential distribution.

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① Exponential distribution.

Def. We say random variable T has exponential distribution with parameter $\lambda > 0$ if

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{o.w} \end{cases}$$

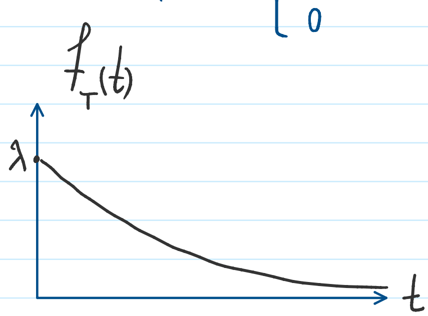
For brevity, we usually write T has $\exp(\lambda)$ distribution.

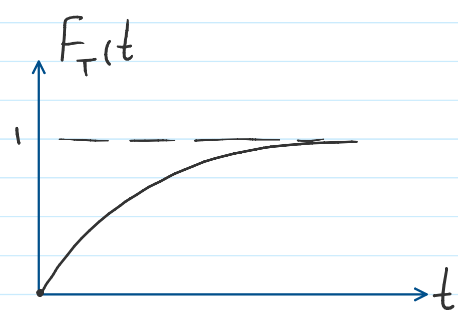
- CDF of T: for $t \geq 0$

$$F_T(t) = \int_{-\infty}^{t} f_T(u)\,du = \int_{-\infty}^{t} \lambda e^{-\lambda u}\,du = -e^{-\lambda u}\Big|_{-\infty}^{t} = 1 - e^{-\lambda t}$$

Hence,

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & \text{o.w.} \end{cases}$$





- complimentary cdf:

$$F^c(t) = P(X > t) = 1 - F(t)$$

- complimentary cdf:

$$F_T^c(t) = P(X > t) = 1 - F_X(t)$$

$$F_T(t) = \begin{cases} e^{-\lambda t} & t \geq 0 \\ 1 & \text{o.w.} \end{cases}$$

- Moments:

$$E[T^n] = \int_0^\infty u^n \lambda e^{-\lambda u} du = -u^n e^{-\lambda u}\Big|_0^\infty + n\int_0^\infty u^{n-1} e^{-\lambda u} du$$

$$= 0 + \frac{n}{\lambda} E[T^{n-1}]$$

- mean, variance and standard deviation:

$$E[T] = \frac{1}{\lambda}$$

$$Var(T) = E[T^2] - (E[T])^2 = \frac{2}{\lambda}\cdot\frac{1}{\lambda} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\sigma_T = \sqrt{Var(T)} = \frac{1}{\lambda}$$

- Memoryless property: notice that $P(T > c) = e^{-\lambda c}$

$$P(T > s+t \mid T > s) = \frac{P(T > s, T > s+t)}{P(T > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t)$$

If T is lifetime of a lamp, given that the lamp has been working for s time units, the probability that it will be working for t additional time units is same as if we replace the lamp with a new one and then look at the probability of this event!

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② geometric distribution vs exponential distribution

Consider the lifetime of a bulb. Let us make the following assumptions:

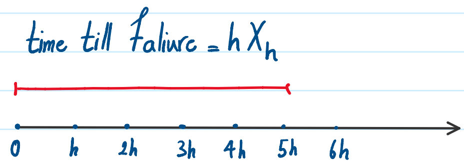
(*) Suppose that we have a clock that ticks every h seconds.

(**) Suppose that the probability that the bulb fails between any two ticks is p_h, independent of everything else, and that $p_h \to 0$ as $h \to 0$.

everything else, and that $p_h \to 0$ as $h \to 0$.

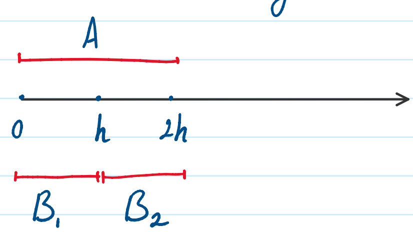
Let X_h denote the number of ticks till faliure. We have,

$$\mathbb{P}(X_h = k) = (1-p_h)^{k-1} p_h :$$



Now suppose we have another clock that ticks every $2h$ seconds. Let p_{2h} denote the probability of faliure of bulb over any interval of length $2h$. Notice that

$$p_{2h} = p_h + p_h - p_h^2 :$$



since

$$P(A) = P(B_1 \cup B_2)$$
$$= P(B_1) + P(B_2) - P(B_1 \cap B_2)$$

A: faliure in interval $[0,2h]$, B_1: faliure in interval $[0,h]$, B_2: faliure in interval $[h,2h]$

Now, if h is small enough, then p_h is small and we have $p_{2h} \approx 2p_h$. Using a simillar argument, and a bit of algebra, we can show that for any $c>0$, $p_{ch} \approx c p_h$ for all h small enough. Hence, the following assumption is not far from being true.

(***) Suppose that $p_h = \lambda h$ for some fixed constant $\lambda > 0$ and any $h > 0$.

Now, given (*), (**), and (***), let us calculate the time it take for the bulb to fail. Notice that if $X_h = k$, then the bulb failed at some time between $(k-1)h$ and kh. Let $T_h = hX_h$ denote the time till tick at which the bulb failed. We have,

$$P(T_h > t) = P(hX_h > t) = P\left(X_h > \frac{t}{h}\right) = (1-p_h)^{\lfloor \frac{t}{h} \rfloor - 1} = (1-\lambda h)^{\lfloor \frac{t}{h} \rfloor - 1}$$

Notice that since X_h is an integer, the following events are the same:

$$\left\{X > \frac{t}{h}\right\} = \left\{X > \left\lfloor \frac{t}{h} \right\rfloor\right\}$$

Now, recall that $\left(1-\frac{x}{a}\right)^a \longrightarrow e^{-x}$ as $a \longrightarrow \infty$

Hence, defining $h = \frac{1}{a}$ we have

$$(1-\lambda h)^{\lfloor \frac{t}{h} \rfloor} = \left(1-\frac{\lambda}{a}\right)^{\lfloor ta \rfloor} = \left(1-\frac{\lambda}{a}\right)^{ta} \cdot \frac{\left(1-\frac{\lambda}{a}\right)^{\lfloor ta \rfloor}}{\left(1-\frac{\lambda}{a}\right)^{ta}}$$

Notice that

(i) $\left(1-\frac{\lambda}{a}\right)^{ta} = \left(\left(1-\frac{\lambda}{a}\right)^a\right)^t \longrightarrow \left(e^{-\lambda}\right)^t = e^{-\lambda t}$ as $a \to \infty$

(ii) $0 < \frac{\left(1-\frac{\lambda}{a}\right)^{\lfloor ta \rfloor}}{\left(1-\frac{\lambda}{a}\right)^{ta}} = \left(1-\frac{\lambda}{a}\right)^{\lfloor ta \rfloor - ta} \leqslant 1,$

and that $\lfloor ta \rfloor - ta \to 0$ as $a \to \infty$; hence $\left(1-\frac{\lambda}{a}\right)^{\lfloor ta \rfloor - ta} \longrightarrow 1$ as $a \to \infty$.

Hence, by above observations:

$$\mathbb{P}(T_h > t) \longrightarrow e^{-\lambda t} \quad \text{as } h \to 0$$

which is the complimentary cdf of an exponential random variable. Hence,

given (*), (**), and (***), the life time of the bulb is an exponential random variable T with parameter λ. In particular, distribution of $T_h = hX_h$ converges to distribution of T as $h \to 0$.