Lecture 17 - 9/30

Monday, October 03, 2022 3:39 AM

Review:

Cummulative distribution function.

Def: cdf of X is denoted by F_X and defined as $F_X(c) = P(X \leq c)$ for any $c \in \mathbb{R}$.

main properties:

F.1. increasing: $a < b \Rightarrow F_X(a) < F_X(b)$

F.2. $\lim_{c \to \infty} F_X(c) = 1$, $\lim_{c \to -\infty} F_X(c) = 0$

F.3. F_X is right continuous, i.e., $\lim_{u \to c^+} F_X(u) = F_X(c)$

Proposition: any function that satisfies above property is cdf of some random variable.

Important implications:

(i) $\Delta F_X(c) = F_X(c) - F_X(c-) = P(X = c)$

(ii) $F_X(c-) = \lim_{u \to c^-} F_X(u) = P(X < c)$

(iii) For any $a < b$, $F_X(b) - F_X(a) = P(a < X \leq b)$

Today:

① Continuous random variable

② Uniform random variable

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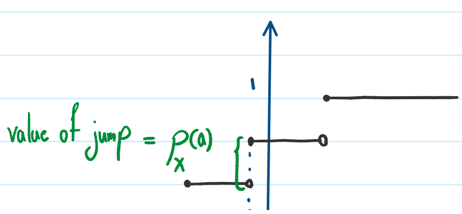
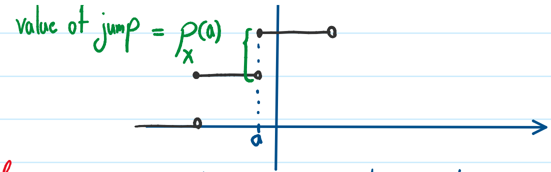
① Continuous random variable

For a discrete random variable, we have $F_X(c) = \sum_{u : u \leq c} p_X(u)$





Def. A random variable X is a continuous type random variable if there exist a function f_X, called probability density function (pdf) of X such that

$$F_X(c) = \int_{-\infty}^{c} f_X(u)\,du. \quad \text{for any } c \in \mathbb{R}$$

Def. support of a pdf f_X is the set of u for which $f_X(u) > 0$.

Important remarks:

(i) Suppose that f_X is continuous at point c. By fundamental theorem of calculus, $f_X(c) = F_X'(c)$

(ii) For a continuous-type random variable X, the cdf F_X is continuous.

(iii) If F_X is continuous at point c, then $F_X(c-) = F_X(c) = F_X(c+) \Rightarrow P(X=c) = \Delta F_X(c) = 0$.

In particular, $P(X=c)=0$ for a continuous-type random variable X, and we have

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$$

(iv) Notice that by axioms of probability, if $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Hence, for any sequence of numbers $x_1, x_2, x_3, \dots$, for a continuous-type random variable X, we have:

$$P\left(X \in \{x_1, x_2, \dots\}\right) = \sum_{i=1}^{n} P(X = x_i) = 0$$

In particular, $P(X \text{ is a natural number}) = 0$

$P(X \text{ is a rational number}) = 0$

Interpretation of pdf:

Suppose that f_X is continuous at point c. We have,

$$\lim_{h \to 0} \frac{F_X(c+h) - F_X(c-h)}{2h} = \lim_{h \to 0} \frac{F_X(c+h) - F_X(c)}{2h} + \frac{F_X(c) - F_X(c-h)}{2h}$$

$$\lim_{h\to 0} \frac{F_X(c+h)-F_X(c-h)}{2h} = \lim_{h\to 0} \frac{F_X(c+h)-F_X(c)}{2h} + \frac{F_X(c)-F_X(c-h)}{2h}$$

$$= \frac{f_X(c)}{2} + \frac{f_X(c)}{2} = f_X(c)$$

where we used the fact that $F_X'(c) = f_X(c)$. Hence, we can write

$$\frac{F_X(c+h)-F_X(c-h)}{2h} = f_X(c) + o(1)$$

where $\lim_{h\to 0} o(1) = 0$. Equivalently, we can write

$$F_X(c+h) - F_X(c-h) = 2h f_X(c) + o(h)$$

where $\lim_{h\to\infty} \frac{o(h)}{h} = 0$, i.e., $o(h)$ converges to zero faster than h.

Hence, $f_X(c)$ is related to the probability of X being in a small neighborhood of c:

$$P(c-h < X < c+h) = 2h f_X(c) + o(h)$$

properties of f_X:

(i) For any $a<b$: $P(a<X\leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du \in [0,1] \Rightarrow f_X$ is non negative.

(ii) $1 = \lim_{b\to+\infty} \lim_{a\to-\infty} F_X(b) - F_X(a) = \lim_{b\to+\infty} \lim_{a\to-\infty} \int_a^b f_X(u)\,du \Rightarrow \int_{-\infty}^{+\infty} f_X(u) = 1$

Mean, variance, LOTUS

We can recycle all we had for discrete random variables by replacing summation with integration, & pmf with pdf, i.e., for a continuous-type random variable X

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} f_X(u)\,du$$

$$Var(X) = E[(X-\mu_X)^2] = \int_{-\infty}^{+\infty} (u-\mu_X)^2 f_X(u)\,du$$ measures how spread out X is

$$Var(X) = E[X^2] - (E[X])^2$$

$\sigma_X = \sqrt{Var(X)} \rightsquigarrow$ Same unit as X

LOTUS: $E[g(X)] = \int_{-\infty}^{+\infty} g(u)\, f_X(u)\, du.$

$$Var(g(X)) = E\left[g(X)^2\right] - \left(E[g(X)]\right)^2$$

$$= \int_{-\infty}^{+\infty} g(u)^2 f_X(u) du - \left(\int_{-\infty}^{\infty} g(u) f_X(u)\, du\right)^2$$

Standardized version: $\frac{X-\mu_X}{\sigma_X} \rightsquigarrow$ dimensionless

and all interpretations are valid as before.

$$E[aX+b] = aE[X]+b$$

$$Var(aX+b) = a^2 Var(X)$$

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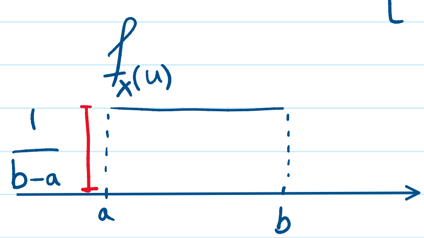
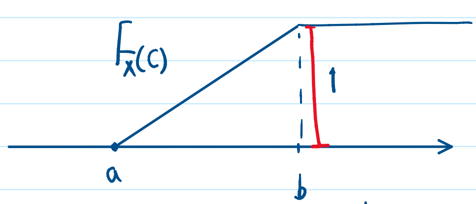
② Uniform random variable:

Def. We say X is uniformly distributed over $[a,b]$ if

$$f_X(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{O.W} \end{cases}$$





kth moment: $E[X^k] = \int_a^b u^k \cdot \frac{1}{b-a} du = \frac{1}{b-a} \cdot \frac{u^{k+1}}{k+1}\Big|_a^b = \frac{b^{k+1} - a^{k+1}}{b-a} \cdot \frac{1}{k+1}$

$\mu_X = E[X] = \frac{b^2-a^2}{1} \cdot \frac{1}{2} = \frac{b+a}{2}$

$$\mu_X = E[X] = \frac{b^2 - a^2}{b-a} \cdot \frac{1}{2} = \frac{b+a}{2}$$

$$\mathrm{var}(X) = E[X^2] - \left(E[X]\right)^2 = \frac{b^3 - a^3}{b-a} \cdot \frac{1}{3} - \frac{(b+a)^2}{4} = \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{(b-a)^2}{12}$$