Lecture 16 - 9/28

Thursday, September 29, 2022 11:31 AM

Review:

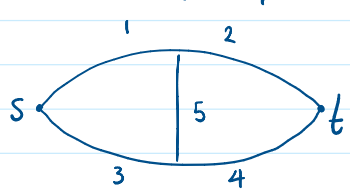
① Union bound:

$P(A \cup B) \leq P(A) + P(B)$, for any $A, B \in \mathcal{F}$

$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)$, for any $A_1, \ldots, A_n \in \mathcal{F}$

② Network outage probability

- We are sending packets from source node to terminal node using an underlying s-t network.
- link i in the s-t network fails with probability $p_i = P(F_i)$, independent of everything else.
- A network outage happens if at least one link fails over each path from s to t.



$\text{Outage} = F_1F_3 \cup F_2F_4 \cup F_1F_4F_5 \cup F_2F_3F_5$

Goal: (i) exactly calculate $P(\text{Outage})$

(ii) use union bound to upperbound $P(\text{Outage})$

③ Distribution of capacity of flow network

- There is a capacity assigned to each link:
  - if link i fails, it cannot pass any packets.
  - if link i works, it can pass packets upto its capacity C_i.

Goal: pmf of capacity of network, i.e., pmf of number of packets that can reach terminal node, from source node.

Today: Cummulative distribution function

Recall definition of probability space $(\Omega, \mathcal{F}, P)$

Ω: sample space, $\mathcal{F}$: set of subsets of Ω, P: probability measure, $P: \mathcal{F} \to [0,1]$

Recall we defined a random variable as a function $X: \Omega \to \mathbb{R}$. To be precise, a a random variable X is a function from Ω to $\mathbb{R}$ such that for any $c \in \mathbb{R}$,

$$\{\omega \in \Omega : X(\omega) \leq c\} \in \mathcal{F}$$

Def: Cummulative distribution function (cdf) of random variable X is denoted by F_X, and is defined as:

$$F_X(c) := P(X \leq c) = P(\{\omega \in \Omega : X(\omega) \leq c\}) \text{ for any } c \in \mathbb{R}$$

Def: Some notations:

$$F_X(c-) := \lim_{\substack{u \to c \\ u < c}} F(u) = \lim_{u \to c-} F(u)$$

$$F_X(c+) := \lim_{\substack{u \to c \\ u > c}} F(u) = \lim_{u \to c+} F(u)$$

$$F_X(+\infty) := \lim_{u \to +\infty} F_X(u)$$

$$F_X(-\infty) := \lim_{u \to -\infty} F_X(u)$$

Properties:

(i) F_X is increasing, i.e. for any $a < b$, we have $F_X(a) \leq F_X(b)$.

proof: $F_X(a) = P(X \leq a) \leq P(X \leq b) = F_X(b)$

where we used axioms of probability & the fact that $\{X \leq a\} \subset \{X \leq b\}$.

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$\boxed{\text{(ii) For any } b > a,\ F_X(b) - F_X(a) = P(a < X \leq b).}$

proof: $F_X(b) - F_X(a) = P(X \leq b) - P(X \leq a) = P(\{X \leq a\} \cup \{a < X \leq b\}) - P(X \leq a) = P(a < X \leq b)$.

$\boxed{\text{(iii) } F_X(+\infty) = 1,\ F_X(-\infty) = 0}$

proof: We will only prove that $F_X(+\infty) = 1$, as the other case is similar.

Consider a sequence $a_1 < a_2 < a_3 < \cdots$ such that $a_n \longrightarrow +\infty$. Let $G_1 = \{X \leq a_1\}$ and $G_k = \{a_k < X \leq a_{k+1}\}$ for all $k \geq 2$. We have,

$$
\begin{aligned}
1 = P(X \in \mathbb{R}) &= P(\{X \leq a_1\} \cup \{a_1 < X \leq a_2\} \cup \{a_2 < X \leq a_3\} \cup \cdots) \\
&= P(X \leq a_1) + P(a_1 < X \leq a_2) + \cdots && \because \text{by axioms of probability} \\
&= P(G_1) + P(G_2) + \cdots \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} P(G_i) = \lim_{n \to \infty} F(a_n). && \because \text{by property (ii)}
\end{aligned}
$$

Hence, $F(+\infty) = 1$.

$\boxed{\text{(iv) } F_X(c-) = P(X < c)}$

proof: Consider a sequence $a_1 < a_2 < a_3 < \cdots < c$ such that $a_n \longrightarrow c$. Let $G_1 = \{X \leq a_1\}$ and $G_k = \{a_k < X \leq a_{k+1}\}$ for all $k \geq 2$. We have,

$$
\begin{aligned}
P(X < c) &= P(\{X \leq a_1\} \cup \{a_1 < X \leq a_2\} \cup \{a_2 < X \leq a_3\} \cup \cdots) \\
&= P(X \leq a_1) + P(a_1 < X \leq a_2) + \cdots && \because \text{by axioms of probability} \\
&= P(G_1) + P(G_2) + \cdots \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} P(G_i) = \lim_{n \to \infty} F(a_n). && \because \text{by property (ii)}
\end{aligned}
$$

Hence, $F_X(c-) = P(X<c)$

$\boxed{\text{(v) } \Delta F_X(c) = F_X(c) - F_X(c-) = P(X=c)}$

proof: $\Delta F_X(c) = P(X\leq c) - P(X<c)$ $\quad\because$ by property (iv)

$\boxed{\text{(vi) } F_X(c^+) = P(X\leq c) = F_X(c) \text{, i.e., } F_X \text{ is right continuous}}$

proof: Consider a sequence $a_1 > a_2 > a_3 > \cdots > c$ such that $a_n \longrightarrow c$.

Notice that for any $w\in\Omega$, $X(w) \leq c$ if and only if $X(w) \leq a_n$ for all n, i.e.,

$$\{X\leq c\} = \bigcap_{i=1}^{\infty}\{X\leq a_i\}$$

Using above set equality, we have

$$F_X(c) = P(X\leq c)$$

$$= P\left(\bigcap_{i=1}^{\infty}\{X\leq a_i\}\right) = 1 - P\left(\bigcup_{i=1}^{\infty}\{X>a_i\}\right) \quad \because \text{Demorgans law}$$

let $G_1 = \{X > a_1\}$ and $G_k = \{a_{k+1} \geq X > a_k\}$ for $k\geq 2$. Using same idea as before,

$$P\left(\bigcup_{i=1}^{\infty}\{X>a_i\}\right) = P(X>a_1) + P(a_1 \geq X > a_2) + \cdots \quad \because \text{axioms of probability}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(G_i)$$

$$= \lim_{n\to\infty} P(G_1 \cup G_2 \cup \cdots \cup G_n) \quad \because \text{axioms of probability}$$

$$= \lim_{n\to\infty} P(X>a_n) = \lim_{n\to\infty} 1 - F_X(a_n)$$

Hence, $F_X(c) = \lim_{n\to\infty} F_X(a_n) = F_X(c+)$.

What functions can be cdf?

Proposition: A function F is the CDF of some random variable if and only if it has

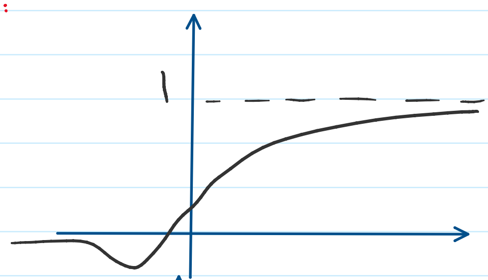
Proposition: A function F is the CDF of some random variable if and only if it has the following properties:

F.1: F is nondecreasing

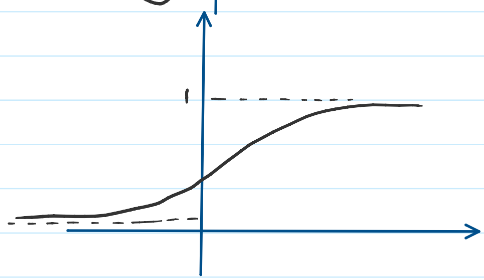
F.2: $F(+\infty) = 1$, $F(-\infty) = 0$

F.3. F is right continuous, i.e., $F_X(c+) = F_X(c)$.

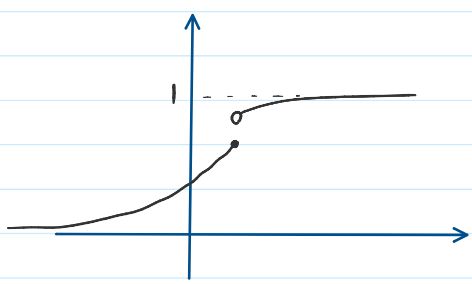
Examples:



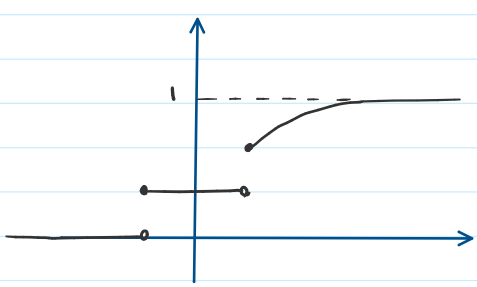
: Not increasing $\Rightarrow$ not cdf



: $F(-\infty) \neq 0 \Rightarrow$ not cdf



: Not right-continuous $\Rightarrow$ not cdf



: Valid cdf