Lecture 15 - 9/26

Monday, September 26, 2022 10:03 AM

Today:

1. Union bound
2. Network outage probability
3. Distribution of capacity of flow network

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1 Union bound

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$, equality if & only if $P(A \cap B) = 0$ (*)
- $P(A_1 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)$, equality if & only if $P(A_i \cap A_j) = 0$ for all i, j

  Notice that by (*), we have $P(A_1 \cup \cdots \cup A_n) \leq P(A_1) + P(A_2 \cup \cdots \cup A_n)$. Repetitive use of this inequality yeilds $P(A_1 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)$
- If $P(A)$ & $P(B)$ are small & A & B are independent, then $P(AB) \ll 1$ and $P(A) + P(B)$ is very close to $P(A \cup B)$

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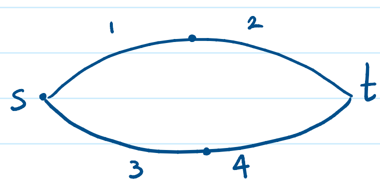
2 Network outage probability

- An s-t network consists of a source s, a terminal node t & bunch of links.
- Link i fails with probability $p_i = P(F_i)$, independent of everything else.

  F_i: event that link i fails.
- Network outage happens if at least one link fails along every s-t path.

Examples:

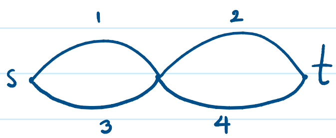
(e1) C.



Outage happens if at least one of $\{1,2\}$ & one of $\{3,4\}$ fail

$$P(F) = P\big((F_1 \cup F_2) \cap (F_3 \cup F_4)\big) = P(F_1 \cup F_2) \cdot P(F_3 \cup F_4)$$
$$= (P_1 + P_2 - P_1P_2)(P_3 + P_4 - P_3P_4)$$

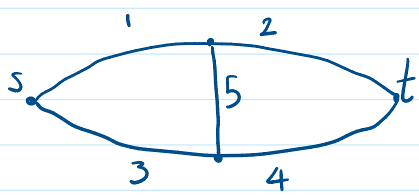
(e2) D:



Outage happens if and only if either $\{1,3\}$ fail or $\{2,4\}$ fail or both.

$$P(F) = P\big((F_1 \cap F_3) \cup (F_2 \cap F_4)\big) = P(F_1 \cap F_3) + P(F_2 \cap F_4) - P(F_1 \cap F_2 \cap F_3 \cap F_4)$$
$$= P_1P_3 + P_2P_4 - P_1P_2P_3P_4$$

(e3) E:



Given link 5 works, E is similar to D & outage happens if either $\{1,3\}$ fail or $\{2,4\}$ fail or both.

Given link 5 fails, E is simillar to C & outage happens if at least one of $\{1,2\}$ & one of $\{3,4\}$ fail.

$$P(F) = P(F_5)P(F \mid F_5) + P(F_5^c)P(F \mid F_5^c)$$

$$= P_5 \cdot (P_1P_3 + P_2P_4 - P_1P_2P_3P_4) + (1 - P_5)(P_1 + P_2 - P_1P_2)(P_3 + P_4 - P_3P_4)$$

We can bound above quantities using union bound

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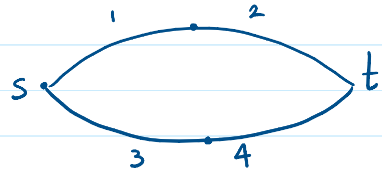
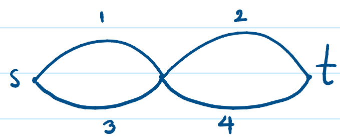
(e1) C.



$$P(F) = P(F_1F_3 \cup F_1F_4 \cup F_2F_3 \cup F_2F_4) \leq P(F_1F_3) + P(F_1F_4) + P(F_2F_3) + P(F_2F_4)$$

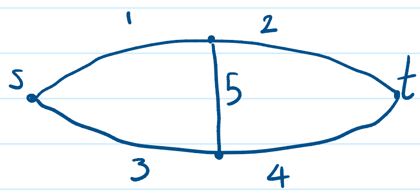
$$= p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4$$

(e2) D.



$$P(F) = P(F_1F_3 \cup F_2F_4) \leq P(F_1F_3) + P(F_2F_4) = p_1p_3 + p_2p_4$$

(e3) E:



$$P(F) = P(F_1F_3 \cup F_2F_4 \cup F_1F_4F_5 \cup F_3F_2F_5)$$

$$\leq P(F_1F_3) + P(F_2F_4) + P(F_1F_4F_5) + P(F_3F_2F_5)$$

$$\leq p_1p_3 + p_2p_4 + p_1p_4p_5 + p_3p_2p_5$$

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③ Distribution of capacity of flow network

Consider a s-t network & suppose that there is a capacity attached to each link. We want to send packages from source node to the terminal node. Link i fails with probability $p_i = P(F_i)$. The capacity of failed link is 0. A link can pass upto its capacity, given it has not failed.
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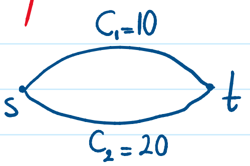
Example:

F:


Possible number of packets that the above system can pass is.

0, 10, 20, 30

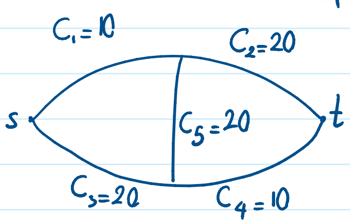
Let X denote the number of packages the system can pass.

$$P_X(0) = P(F_1 F_2) = p_1 p_2$$

$$P_X(10) = P(F_1^c F_2) = (1-p_1)p_1$$

$$P_X(20) = P(F_1 F_2^c) = p_1(1-p_2)$$

$$P_X(30) = P(F_1^c F_2^c) = (1-p_1)(1-p_2)$$

E.


Possible number of packets that the above system can pass is.

0, 10, 20, 30

Let X denote the number of packages the system can pass.

$$P_X(0) = P(\text{network outage})$$

$$= p_5 \cdot (p_1 p_3 + p_2 p_4 - p_1 p_2 p_3 p_4) + (1-p_5)(p_1 + p_2 - p_1 p_2)(p_3 + p_4 - p_3 p_4)$$

$$P_X(30) = P(F_1^c F_3^c F_2^c F_4^c F_5^c) = (1-p_1)(1-p_2)(1-p_3)(1-p_4)(1-p_5)$$

$P_X(30) = P(F_1\ F_3\ F_2\ F_4\ F_5) = (1-p_1)(1-p_2)(1-p_3)(1-p_4)(1-p_5)$

For passing 30 packets, all links should work.

$$P_X(20) = P(F_3^c F_2^c F_5^c F_1 \cup F_3^c F_2^c F_5^c F_4 \cup F_5 F_1^c F_2^c F_3^c F_4^c)$$

$$= P(F_3^c F_2^c F_5^c F_1 \cup F_3^c F_2^c F_5^c F_4) + P(F_5 F_1^c F_2^c F_3^c F_4^c)$$

$$= p_5(1-p_1)(1-p_2)(1-p_3)(1-p_4) + (1-p_3)(1-p_2)(1-p_5)(p_1+p_4-p_1p_4)$$

Notice that for passing 20 packets, link 2 & 3 should work. Now if link 5 works then either $\{1,4\}$ should fail, otherwise we can pass 30. If link 5 fails, then both $\{1,4\}$ should work so to pass 20 packets.