Lecture 14 - 9/23

Tuesday, September 20, 2022 9:37 AM

ML vs MAP:

①. ML maximizes likelihood of observation, MAP maximizes a posteriori probabilites

ML: Given $X=k$, $\begin{cases} \text{declare } H_0 \text{ if } P(X=k|H_0) > P(X=k|H_1) \\ \text{declare } H_1 \text{ if } P(X=k|H_0) < P(X=k|H_1) \\ \text{declare either if } P(X=k|H_0) = P(X=k|H_1) \end{cases}$

MAP: Given $X=k$ $\begin{cases} \text{declares } H_0 \text{ if } P(H_0|X=k) > P(H_1|X=k) \\ \text{declares } H_1 \text{ if } P(H_0|X=k) < P(H_1|X=k) \\ \text{declares eithe if } P(H_0|X=k) = P(H_1|X=k) \end{cases}$

②. MAP rule maximizes joint probability of system and observation:

$P(H_0|X=k) > P(H_1|X=k)$ if and only if $P(H_0, X=k) > P(H_1, X=k)$

$P(H_0|X=k) < P(H_1|X=k)$ if and only if $P(H_0, X=k) < P(H_1, X=k)$

③. For ML decision rule, we defined liklihood matrix.

	$X=1$	$X=2$	$X=3$	
H_0	$P(X=1\|H_0)$	$P(X=2\|H_0)$	$P(X=3\|H_0)$	⇝ pmf of X given H_0 is true, row sum is one
H_1	$P(X=1\|H_1)$	$P(X=2\|H_1)$	$P(X=3\|H_1)$	⇝ pmf of X given H_1 is true, row sum is one

ML decision rule is to pick largest value from each column in likelihood matrix.

	$X=1$	$X=2$	$X=3$	
$H=0$	$P(X=1\|H_0)$	$\boxed{P(X=2\|H_0)}$	$P(X=3\|H_0)$	⇝ pick H_0 if $X=2$
$H=1$	$\boxed{P(X=1\|H_1)}$	$P(X=2\|H_1)$	$\boxed{P(X=3\|H_1)}$	⇝ pick H_1 if $X=1$ or 3

④. For MAP, we define joint Probability matrix

$\boxed{\text{note: } \pi_0 + \pi_1 = 1}$

	$X=1$	$X=2$	$X=3$	
H_0	$P(X=1, H_0)$	$P(X=2, H_0)$	$P(X=3, H_0)$	⇝ joint distribution of X & H_0, row sum is π_0
H_1	$P(X=1, H_1)$	$P(X=2, H_1)$	$P(X=3, H_1)$	⇝ joint distribution of X & H_1, row sum is π_1

H_0	$P(X=1,H_0)$	$P(X=2,H_0)$	$P(X=3,H_0)$	$\rightsquigarrow$ joint distribution of X & H_0, row sum is π_0
H_1	$P(X=1,H_1)$	$P(X=2,H_1)$	$P(X=3,H_1)$	$\rightsquigarrow$ joint distribution of X & H_1, row sum is π_1

MAP decision rule is to pick largest value from each column in joint probability matrix

	$X=1$	$X=2$	$X=3$	
H_0	$P(X=1,H_0)$	$\boxed{P(X=2,H_0)}$	$\boxed{P(X=3,H_0)}$	$\rightsquigarrow$ Pick H_0 if $X=2$ or 3
H_1	$\boxed{P(X=1,H_1)}$	$P(X=2,H_1)$	$P(X=3,H_1)$	$\rightsquigarrow$ Pick H_1 if $X=1$

⑤ In ML, we compare likelihood ratio, $\Lambda(k) = \dfrac{p_1(k)}{p_0(k)} = \dfrac{P(X=k|H_1)}{P(X=k|H_0)}$ with 1.

$$\text{ML}: \Lambda(k): \begin{cases} \text{declare } H_0 \text{ if } & \Lambda(k) < 1 \\ \text{declare } H_1 \text{ if } & \Lambda(k) > 1 \\ \text{declare either if } & \Lambda(k) = 1 \end{cases}$$

equivalently, we compare $\log \Lambda(k)$ with 0.

$$\text{ML}: \log \Lambda(k) = \log p_1(k) - \log p_0(k): \begin{cases} \text{declare } H_0 \text{ if } & \log \Lambda(k) < 0 \\ \text{declare } H_1 \text{ if } & \log \Lambda(k) > 0 \\ \text{declare either if } & \log \Lambda(k) = 0 \end{cases}$$

⑥ In MAP, we compare likelihood ratio, $\Lambda(k) = \dfrac{p_1(k)}{p_0(k)} = \dfrac{P(X=k|H_1)}{P(X=k|H_0)}$ with $\dfrac{\pi_0}{\pi_1}$.

$$\text{MAP}: \Lambda(k): \begin{cases} \text{declare } H_0 \text{ if } & \Lambda(k) < \frac{\pi_0}{\pi_1} \\ \text{declare } H_1 \text{ if } & \Lambda(k) > \frac{\pi_0}{\pi_1} \\ \text{declare either if } & \Lambda(k) = \frac{\pi_0}{\pi_1} \end{cases}$$

equivalently, we compare $\log \Lambda(k)$ with $\log\left(\frac{\pi_0}{\pi_1}\right)$

$$\text{MAP}: \log \Lambda(k) = \log p_1(k) - \log p_0(k): \begin{cases} \text{declare } H_0 \text{ if } & \log \Lambda(k) < \log \pi_0 - \log \pi_1 \\ \text{declare } H_1 \text{ if } & \log \Lambda(k) > \log \pi_0 - \log \pi_1 \\ \text{declare either if } & \log \Lambda(k) = \log \pi_0 - \log \pi_1 \end{cases}$$

⑦. If $\pi_0 = \pi_1 = 1/2$, then MAP and ML are equivalent.

Important definitions.

$P_{\text{false alarm}}$: probability that decision rule decides H_1, given true state is H_0 ⤳ conditional prob.

P_{miss}: probability that decision rule decides H_0, given true state is H_1 ⤳ conditional prob.

P_{error}: probability that decision rule is not equal to the true state. ⤳ regular probability.

Let D denote the decision rule & S denote the true state of the system:

$$P_{\text{false alarm}} = P(D=H_1 \mid S=H_0)$$

$$P_{\text{miss}} = P(D=H_0 \mid S=H_1)$$

$$P_{\text{error}} = P(D \neq S) = P(D=H_0 \cap S=H_1) + P(D=H_1 \cap S=H_0)$$

$$= P(S=H_0) \cdot P(D=H_1 \mid S=H_0) + P(S=H_1) P(D=H_0 \mid S=H_1)$$

$$= \pi_0 \cdot P_{\text{false alarm}} + \pi_1 P_{\text{miss}}$$

P_{error} is the sum of elements in joint probability matrix that are not selected. In above example: $P_{\text{error}} = P(X=1, H_0) + P(X=2, H_1) + P(X=3 \mid H_1)$

Hence, MAP minimizes P_{error}.

Examples:

Problem 4 (Monty Hall) Participants in a game show are asked to choose one of three doors. Behind one of the doors, there is a prize, the other doors lead to empty rooms. After the door is chosen, the game show presenter (who knows the right door) opens an unchosen door which is empty and shows it to the participant. The participant is then given a choice to change his decision. Should he do it?

solution: Without loss of generality, assume we pick door 1. Define,

H_0 : the reward is behind door one

H_1 : the reward is behind door two or three

Since we have no idea where the reward is, a reasonable choice for π_0 & π_1 is

$$\pi_0 = P(H_0) = 1/3 \;,\; \pi_1 = P(H_1) = 2/3$$

Suppose that the presenter randomly opens an empty door. Let X denote the door which is opened.

Given H_0, $p_0(2) = p_0(3) = 1/2$

Given H_1, the reward is behind door 2 with probability $\frac{1}{2}$ & behind door 3 with probability $\frac{1}{2}$, and presenter has to open an empty door; hence, $p_1(2) = p_1(3) = 1/2$

Suppose that we observed door 2 is opened:

ML. $P(X=2|H_0) = 1/2$, $P(X=2|H_1) = 1/2 \Rightarrow \Lambda(2) = \frac{1/2}{1/2} = 1$

$\Rightarrow$ ML is indecisive

MAP: $\Lambda(2) = 1 > \frac{\pi_0}{\pi_1} = \frac{1}{2}$

$\Rightarrow$ MAP decides to switch doors.

Important remark:

The problem asks for best decision rule, i.e, the one that maximizes the probability of winning after the presenter opens a door. i.e., after the observation. This is the MAP decision rule since MAP minimizes P_{error}

Consider an optical communication system that transmits a symbol "1" by turning a laser on for 200 nanoseconds, and a "0" by turning the laser off for 200 nanoseconds. A photon counting device counts the number of photons in the 200 nanosecond time interval (synchronized with the transmitting laser). Assume that the number of photons measured within a given interval (200 nanoseconds) is a Poisson random variable with parameters $\lambda_1 = 5$ and $\lambda_0 = 1$, depending on whether the laser is on or off. Note that even when the laser is off, it still emits a random number of photons due to the dark current. Also assume "0" is 5 times as likely as "1" to be transmitted. Suppose that the photon counting device detects 5 photons in a given time interval.

(i) What is MAP decision rule for the observation?

(ii) What is the probability that "0" was sent?

(iii) What is the probability of error for MAP?

(iii) What is the probability of error for MAP?

Solution. Let X denote # of recieved photons.

(i) $\frac{\pi_0}{\pi_1} = 5 \Rightarrow \ln\left(\frac{\pi_0}{\pi_1}\right) = \ln(5)$

$$\mathcal{L}(k) = \frac{P_1(k)}{P_0(k)} = \frac{e^{-5}\, 5^k}{e^{-1}} = e^{-4}\, 5^k \Rightarrow \ln(\mathcal{L}(k)) = -4 + k \ln 5$$

Given $X = 5$, $-4 + 5\ln 5 > \ln 5$ since $\ln 5 > 1 \Rightarrow$ MAP decision rule is "1" was sent

(ii) $\pi_0 + \pi_1 = 1 \quad \Rightarrow 5\pi_1 + \pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{6}$ and $\pi_0 = \frac{5}{6}$

$$P(\text{"0" was sent} \mid X=5) = \frac{P(X=5 \mid \text{"0" was sent}) \cdot P(\text{"0" was sent})}{P(X=5)}$$

$$= \frac{\frac{5}{6} \times e^{-1} \frac{1}{5!}}{\frac{5}{6} \times e^{-1} \frac{1}{5!} + \frac{1}{6} \times e^{-5} \frac{5^5}{5!}} = \frac{1}{e^{-4} \times 5^4}$$

(iii) Comparing $\log \mathcal{L}(k)$ with $\log\left(\frac{\pi_0}{\pi_1}\right)$, we have

$$\text{MAP}: \begin{cases} \text{"0" was sent}, & \text{if } X \leq 3 \\ \text{"1" was sent}, & \text{if } X \geq 4 \end{cases}$$

$$P_{\text{false alarm}} = P(X \geq 4 \mid \text{"0" was sent}) = \sum_{k=4}^{\infty} \frac{e^{-1}}{k!}$$

$$P_{\text{miss}} = P(X \leq 3 \mid \text{"1" was sent}) = \sum_{k=0}^{3} e^{-5} \frac{5^k}{k!}$$

$$P_{\text{error}} = \pi_0 \cdot P_{\text{false alarm}} + \pi_1 \cdot P_{\text{miss}}$$