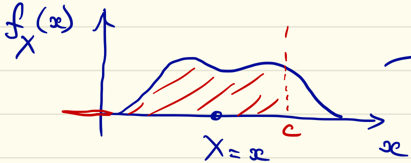
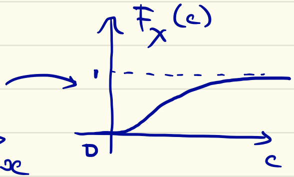
ECE 313: Lecture 20
Uniform distribution
Exponential distribution

Recall: continous-type rv (rand varible)

$$X \sim \underset{\uparrow \text{ pdf}}{f_X}$$





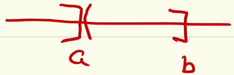
CDF: $$P\{X \le c\} = \int_{-\infty}^{c} f_X(x)\,dx = F_X(c)$$

Hence $$f_X(x) = F_X'(x) = \frac{dF_X(c)}{dc}$$

$$P\underbrace{\{a < X \le b\}}_{B \setminus A} = P\underbrace{\{X \le b\}}_{B} - P\underbrace{\{X \le a\}}_{A}$$

$$= F_X(b) - F_X(a)$$

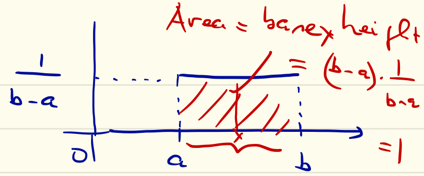
Because $\begin{cases} A \cup (B \setminus A) = B \\ A \cap (B \setminus A) = \emptyset \end{cases}$



$*$ $$P\{X = a\} = P\{X \le a\} - P\{X < a\} = F_X(a) - F_X(a-) = 0$$

for $F_X(x)$ that is continuous at $x = a$

Example 1: Uniform r.v.



$$U \sim \text{Unif}[a, b]$$

$$\underbrace{f_U(u)}_{\uparrow\ \text{pdf}} = \begin{cases} \frac{1}{b-a} & \text{for } a \leq u \leq b \\ 0 & \text{else} \end{cases}$$

$$\underset{\underset{1}{\parallel}}{P\{U \leq +\infty\}} = \int_{-\infty}^{+\infty} f_U(u)\, du = \int_a^b \frac{1}{b-a}\, du = \frac{1}{b-a} (u) \Big|_a^b$$

$$= \frac{1}{b-a}(b-a) = 1$$

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_{-\infty}^{+\infty} u\, f_X(u)\, du$$

$$E[\underbrace{g(X)}_{Y}] \underset{\uparrow\ \text{LOTUS}}{=} \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx \qquad X = x$$

$\underset{\sim}{\text{Ex}}$: (Unif.)

$$f_U(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{else} \end{cases}$$

$$E[U] = \int_a^b u \cdot \underbrace{\frac{1}{b-a}}_{f_U(u)} \, du = \frac{1}{b-a} \left(\frac{u^2}{2}\right)\Bigg|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^2-a^2}{2} = \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)} \cdot 2}$$

$$\boxed{\text{Note : } (u^n)' = n u^{n-1}}$$

$$\mathrm{Var}[X] \stackrel{\text{def}}{=} E\Big[\big(X - \underbrace{E[X]}_{\mu_X}\big)^2\Big] = E\left[(X-\mu_X)^2\right]$$

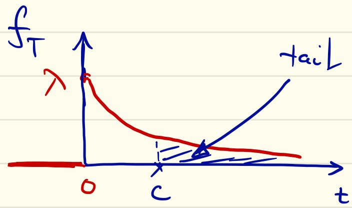
$$= E\left[X^2 - 2\mu_X X + \mu_X^2\right]$$

$$\stackrel{\text{Linear prop. of expectation}}{=} E[X^2] - 2\mu_X E[X] + \mu_X^2$$

$$= E[X^2] - (E[X])^2$$

(Unif cont.)

$$E[U^2] = \int_a^b u^2 \cdot \underbrace{\frac{1}{b-a}}_{f_U(u)} \, du = \dots$$



⊛ <u>Example</u> 2 : Exponential R.V. $(\lambda > 0)$

$$T \sim Exp(\lambda)$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$= 1 - P\{T \leq c\}$$
$$= 1 - F_T(c)$$
$$= P\{T > c\}$$

$$= \int_c^{+\infty} f_T(t) \, dt = \int_c^{+\infty} \lambda e^{-\lambda t} \, dt$$

$$= \left(-e^{-\lambda t}\right)\Big|_c^{+\infty} = e^{-\lambda c}$$

$$\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$$

3.7. **[Using an exponential distribution]**
Let T be an exponentially distributed random variable with parameter $\lambda = \ln 2$.

(a) Find the simplest expression possible for $P\{T \geq t\}$ as a function of t.

(b) Find $P(T \leq 1 | T \leq 2)$.

Solutions

(a)
$$P\{T \geq t\} = e^{-\lambda t} = \left(e^{\ln 2}\right)^{-t} = 2^{-t}$$

(b)
$$P(T \leq 1 \mid T \leq 2) = \frac{P(T \leq 1,\ T \leq 2)}{P(T \leq 2)}$$

$$= \frac{P(T \leq 1)}{P(T \leq 2)} = \frac{1 - P(T \geq 1)}{1 - P(T \geq 2)} = \frac{1 - 2^{-1}}{1 - 2^{-2}}$$