University of Illinois Fall 2019

ECE 313: Final Exam

Monday, December 16, 2019
7:00 p.m. — 10:00 p.m.

1. **[5+5+5 points]** Consider a game with probability of winning $p = 1/3$. If we win, we receive \$10, otherwise we pay \$2. Assume that we play the game until we win for the first time.

   (a) Find the probability that we earn \$4.
   Solution: To earn \$4, we need to lose 3 times (we lose \$2 every time) and then to win one time (we win \$10). The probability of this event is $(2/3) \times (2/3) \times (2/3) \times (1/3) = 8/81$.

   (b) Find the mean value of our payoff (i.e., our expected earnings).
   Solution: Let L denote the number of games played until we win the game for the first time. Then, $L \sim \text{Geo}(1/3)$. Our payoff will be $-2(L-1)+10$. Taking the expectation and using the fact that $E[L] = 1/p = 3$, our expected payoff is $-2(3-1)+10 = 6$.

   (c) Now assume that instead of stopping at the first win, we keep playing the game an infinite number of times. Let L_1 be the number of games needed until the first win. Let L_2 denote the number of games, after the first L_1 trials, until the second win. Find $\mathbb{P}(L_2 = 3 | L_1 = 5)$.
   Solution: L_1 is clearly independent of L_2 (in a Bernoulli process, all geometric random variables are independent). Therefore, $\mathbb{P}(L_2 = 3|L_1 = 5) = \mathbb{P}(L_2 = 3) = (2/3) \times (2/3) \times (1/3) = 4/27$.

2. **[6+6+6+3 points]** Consider the experiment of rolling two fair dice, each with 6 faces numbered 1, 2, 3, 4, 5, 6. Let S and P denote the sum and product of the numbers showing on the two dice, respectively.

   (a) Find the mean of P.
   Solution: Let X_1 and X_2 denote the numbers showing on the two dice. We have

   $$P\{X_i = k\} = \frac{1}{6}, \quad \text{for } i = 1, 2;\ k = 1, 2, \ldots, 6.$$

   Furthermore, X_1 and X_2 are independent. The mean of P can be calculated as

   $$E[P] = E[X_1 X_2] = E[X_1]E[X_2],$$

   while for $i = 1, 2$ we have

   $$E[X_i] = \sum_{k=1}^{6} \frac{1}{6} k = \frac{7}{2}.$$

   Hence,

   $$E[P] = \left(\frac{7}{2}\right)^2 = \frac{49}{4}.$$

   (b) Find the probability that S is even.
   Solution: Define the following events: E_i = "X_i is even", O_i = "X_i is odd" for $i = 1, 2$. Clearly, $P(E_i) = P(O_i) = 1/2, i = 1, 2$. We have $P(\text{"}S \text{ is even"}) = P(E_1E_2) + P(O_1O_2) = P(E_1)P(E_2) + P(O_1)P(O_2)$, since X_1 and X_2 are independent. Thus,

   $$P(\text{"}S\text{ is even"}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

(c) Find the probability that S is even given that P is even.
Solution: We have

$$P(\text{“}S\text{ is even”}|\text{“}P\text{ is even”}) = \frac{P(\text{“}S\text{ is even”}, \text{“}P\text{ is even”})}{P(\text{“}P\text{ is even”})} = \frac{P(E_1E_2)}{1 - P(O_1O_2)} = \frac{1}{3}.$$

(d) Are “S is even” and “P is even” mutually independent? Justify your answer.
Solution: Since $P(\text{“}S\text{ is even”}) \neq P(\text{“}S\text{ is even”}|\text{“}P\text{ is even”})$, “$S$ is even” and “P is even” are *not* mutually independent.

3. **[6+6+6 points]** When looking for the next smartphone to buy, you narrow down to two leading brands. The first brand claims that their phone lifetime is uniformly distributed over the interval 0 to 4 years, while the second brand claims that their phone lifetime in years is exponentially distributed with parameter $\lambda = 1/2$.

(a) If you use the expectation of lifetime (the larger the better), then which brand should you pick? Justify your answer.
Solution: Let X be the lifetime of a phone from the first brand and Y be the lifetime of a phone from the second brand. We have $X \sim \text{Unif}[0,4]$ and $Y \sim \text{Exp}(\lambda)$, where $\lambda = 1/2$. Therefore, $E[X] = (0+4)/2 = 2$ and $E[Y] = 1/\lambda = 2$. Hence, both brands are equally good in terms of expectation of lifetime.

(b) If you will replace your phone after 2 years anyway, then a better metric would be the probability that the phone is still working after 2 years. Which brand should you pick now? Justify your answer.
Solution: Using the pdf and properties of uniform and exponential random variables we have

$$P(X > 2) = \int_2^4 \frac{1}{4} du = \frac{1}{2},$$
$$P(Y > 2) = e^{-\lambda 2} = e^{-1} = \frac{1}{e}.$$

Hence, $P(X > 2) > P(Y > 2)$, i.e., you should pick the first brand.

(c) If your mother only wants to replace her phone after 5 years, then which brand would you pick for her? Justify your answer.
Solution: Only phones from the second brand have positive probability of working after five years. Hence, you should choose the second brand for your mother.

4. **[4+6+4 points]** Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 4$ per hour. Let N_t denote the number of buses arriving in the time interval $[0,t]$. Recall that for a fixed $t > 0$, N_t is a Poisson random variable with parameter $4t$.

(a) Find the probability that no bus arrives in the first $t = 0.25$ hours. Provide your answer in terms of e.
Solution: The probability is given by $\mathbb{P}[N_{0.25} = 0] = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = \frac{e^{-4\times 0.25}(4\times 0.25)^0}{0!} = e^{-1}$.

(b) Find the conditional probability that there is 1 arrival in the interval $(0.5h, 1h]$ given that there are 2 arrivals in the interval $[0, 1h]$. Here, ‘h’ denotes ‘hours’.
Solution: Suppressing ‘h’ for notational convenience, let A be the event that there is 1 arrival in the time interval $(0.5, 1]$, B the event that there are 2 arrivals in the interval $[0,1]$ and C the event that there is 1 arrival in the interval $[0, 0.5]$. Then, the conditional

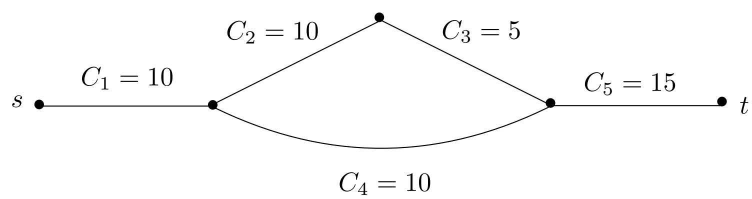
probability is given by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A \cap C]}{\mathbb{P}[B]} \underbrace{=}_{\text{ind. incr. property}} \frac{\mathbb{P}[A]\mathbb{P}[C]}{\mathbb{P}[B]} = \frac{\mathbb{P}[N_{0.5} = 1]\mathbb{P}[N_{0.5} = 1]}{\mathbb{P}[N_1 = 2]}$$

$$= \frac{\frac{e^{-4\times 0.5}(4\times 0.5)^1}{1!} \frac{e^{-4\times 0.5}(4\times 0.5)^1}{1!}}{\frac{e^{-4\times 1}(4\times 1)^2}{2!}} = \frac{\frac{e^{-2}2^1}{1!} \frac{e^{-2}2^1}{1!}}{\frac{e^{-4}4^2}{2!}} = \frac{4e^{-2}e^{-2}}{e^{-4}8} = \frac{1}{2}.$$

(c) Let X denote the number of arrivals in $[0, 1h]$ and let Y denote the number of arrivals in $(1h, 2h]$. Find $\mathbb{P}(Y = 2|X = 1)$.
Solution: Since X and Y are independent and X and Y have identical distributions, $\mathbb{P}[Y = 2|X = 1] = \mathbb{P}[Y = 2] = \mathbb{P}[N_1 = 2] = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \frac{e^{-4}(4)^2}{2!} = 8e^{-4}$.

5. **[4+4+6 points]** Consider the following $s - t$ network, where link i fails independently with probability p_i:



Denote by $q_i = 1 - p_i$ the probability that link i works.

(a) Let Y denote the capacity of the network, i.e., the maximum flow rate from s to t. What are the possible values of Y?
Solution: Y takes values in the set $\{0, 5, 10\}$.

(b) Compute $P(Y = 5)$.
Solution: $Y = 5$ if all links work except for link 4. Therefore, $P(Y = 5) = q_1q_2q_3p_4q_5$.

(c) Compute the probability of network outage, which corresponds to the event that at least one link fails along every $s - t$ path.
Solution: The network fails if either link 1 or 5 fail, which happens with probability $p_1 + p_5 - p_1p_5$.
If links 1 and 5 work, then the network fails if both link 4 and the serial link $2 - 3$ fails, which has probability $q_1q_5p_4p_{2,3}$. Here, $p_{2,3}$ denotes the probability that the serial link $2 - 3$ fails, which is given by $p_{2,3} = p_2 + p_3 - p_2p_3$.
Therefore, we have

$$P(\text{outage}) = P(Y = 0) = p_1 + p_5 - p_1p_5 + q_1q_5p_4(p_2 + p_3 - p_2p_3).$$

6. **[12+12 points]** The two parts of this problem are unrelated.

(a) A blind man waits at a bus stop serviced by the buses A and B. He plans to take the next bus arriving at the bus stop. Let X denote the arrival time of bus A and Y denote the arrival time of bus B. X is an exponential random variable with mean value 1 and Y is also exponential with mean value 10. Additionally, X and Y are independent. The blind man wants to take bus A. What is the probability that he takes the wrong bus?
Solution: He takes the wrong bus when Y is less than X. X and Y are independent so the joint distribution is the product of the marginals.

$$
\begin{aligned}
P(Y < X) &= \int_0^\infty \int_v^\infty f_{X,Y}(u,v)dudv \\
&= \int_0^\infty \int_v^\infty e^{-u}(0.1e^{-0.1v})dudv \\
&= \int_0^\infty e^{-v}(0.1e^{-0.1v})dv \\
&= \int_0^\infty 0.1e^{-1.1v}dv \\
&= 1/11
\end{aligned}
$$

(b) Let X and Y be random variables with joint pdf

$$
f_{X,Y}(u,v) = \begin{cases} 8uv, & 0 \le u \le v, 0 \le v \le 1 \\ 0, & \text{otherwise} \end{cases}
$$

Find $f_{X|Y}(u|v)$ for any $0 \le u \le v \le 1$ and $E[X|Y=v]$ for any $0 \le v \le 1$.

Solution: We first note that $f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)}$.

$$
f_Y(v) = \int_0^v 8uvdu = 4vu^2|_0^v = 4v^3, \quad 0 \le v \le 1
$$

$$
f_{X|Y}(u|v) = \frac{8uv}{4v^3} = \frac{2u}{v^2}, \quad 0 \le u \le v \le 1.
$$

Moreover,

$$
\begin{aligned}
E[X|Y=v] &= \int_0^v u f_{X|Y}(u|v)du \\
&= \int_0^v \frac{2u^2}{v^2}du \\
&= \frac{2v}{3}, \quad 0 \le v \le 1.
\end{aligned}
$$

7. **[7+7+7 points]** Let $R_1 = 1 + W_1$ denote the value of a 1Ω resistor, where $W_1 \sim \text{Unif}[-1,1]$ is the manufacturing error. Let $R_2 = 2 + W_2$ denote the value of a 2Ω resistor, where $W_2 \sim \text{Unif}[-1,1]$ is the manufacturing error as well. Assume that W_1 and W_2 are independent, i.e., R_1, R_2 are independent. Suppose that a 3Ω resistor is made by concatenating R_1 and R_2, i.e., $R_3 = R_1 + R_2$.

(a) Find $E[R_3]$ and $\text{Var}(R_3)$.

Solution: Since $R_3 = R_1 + R_2 = 3 + W_1 + W_2$, the mean is given by $\mathbb{E}[R_3] = \mathbb{E}[3 + W_1 + W_2] = 3$. The variance is given by

$$
\text{Var}(R_3) = \text{Var}(3 + W_1 + W_2) = \text{Var}(W_1) + \text{Var}(W_2) = \frac{2^2}{12} + \frac{2^2}{12} = \frac{2}{3}.
$$

Here, the independence of W_1 and W_2 has been used.

(b) Assume that we bought 10 samples $X_1, X_2, \ldots, X_{10}$, of R_1, i.e., $X_i = 1 + W_i$ and $W_i \sim \text{Unif}[-1, 1], i = 1, 2, \ldots, 10$ are independent random variables. Find the mean square error, $\mathbb{E}[(\hat{X} - \mu)^2]$, of the sample mean $\hat{X} = (X_1 + X_2 + \cdots + X_{10})/10$, where $\mu = E[X_i], i = 1, 2, \ldots, 10$.

Solution: First, $\sigma^2 = \text{Var}(X_i) = \text{Var}(W_i) = \frac{2^2}{12} = \frac{1}{3}, i = 1, 2, \ldots, 10$. Additionally, $\hat{X}$ is unbiased, i.e., $E[\hat{X}] = \mu = 1$. Then, the MSE is

$$\mathbb{E}\left[(\hat{X} - \mu)^2\right] = \text{Var}(\hat{\text{X}}) = \frac{1}{100}\sum_{i=1}^{10} \text{Var}(X_i) = \frac{\sigma^2}{10} = \frac{1}{30}.$$

(c) Use Markov's inequality to upper bound $\mathbb{P}\left((\hat{X} - \mu)^2 \geq 0.1\right)$.

Solution: By Markov's inequality,

$$\mathbb{P}\left[(\hat{X} - \mu)^2 \geq 0.1\right] \leq \frac{\mathbb{E}\left[(\hat{X} - \mu)^2\right]}{0.1} = 10 \times \frac{1}{30} = \frac{1}{3}.$$

8. **[10+10+10 points]** Assume that if hypothesis 0 (H_0) is true, then the random variable X takes values $-2, -1, 0, 1, 2$, each with probability $1/5$, and if hypothesis 1 (H_1) is true, then the random variable X takes the values -1 with probability $1/4$, 0 with probability $1/2$ and 1 with probability $1/4$. The prior probabilities satisfy $\pi_0/\pi_1 = 2$.

(a) Find the MAP decision rule given an observation $X = k$.
Solution:

X	-2	-1	0	1	2
H_0	1/5	1/5	1/5	1/5	1/5
H_1	0	1/4	1/2	1/4	0

It is clear that for $X = 2$ and $X = -2$, H_0 will be selected.
For $X = -1$ and $X = 1$, we have $\Lambda(1) = \Lambda(-1) = \frac{1/4}{1/5} = \frac{5}{4} < 2$ and hence, H_0 will be selected as well.
For $X = 0$, we have $\Lambda(0) = \frac{1/2}{1/5} = \frac{5}{2} > 2$ and hence, H_1 will be selected in this case.

(b) Compute the average error probability p_e of the MAP decision rule.
Solution: From $\pi_0/\pi_1 = 2$ we get $\pi_0 = 2/3$ and $\pi_1 = 1/3$.

$$\begin{aligned} p_e &= \pi_0 p_{false\,alarm} + \pi_1 p_{miss} \\ &= \frac{2}{3}\frac{1}{5} + \frac{1}{3}\left(\frac{1}{4} + \frac{1}{4}\right) \\ &= \frac{27}{90} \end{aligned}$$

(c) Suppose that instead of an observation of X we are given the sum of two independent realizations of X (under the same hypothesis). If the sum of these two realizations is 0, which hypothesis will the ML decision rule declare as the true hypothesis?
Solution:
Denote by X_1 and X_2 the outcome of the two realizations of X, and by Y the sum $X_1 + X_2$.

Under H_0, we have

$$\begin{aligned} P(Y=0|H_0) &= P(X_1=0, X_2=0|H_0) + P(X_1=1, X_2=-1|H_0) \\ &+ P(X_1=-1, X_2=1|H_0) + P(X_1=-2, X_2=2|H_0) \\ &+ P(X_1=2, X_2=-2|H_0) \\ &= P(X_1=0|H_0)P(X_2=0|H_0) + P(X_1=1|H_0)P(X_2=-1|H_0) \\ &+ P(X_1=-1|H_0)P(X_2=1|H_0) + P(X_1=-2|H_0)P(X_2=2|H_0) \\ &+ P(X_1=2|H_0)P(X_2=-2|H_0) = 5\left(\frac{1}{5}\right)^2 = \frac{1}{5}. \end{aligned}$$

Under H_1, we have

$$\begin{aligned} P(Y=0|H_1) &= P(X_1=0, X_2=0|H_1) + P(X_1=1, X_2=-1|H_1) + P(X_1=-1, X_2=1|H_1) \\ &= P(X_1=0|H_1)P(X_2=0|H_1) + P(X_1=1|H_1)P(X_2=-1|H_1) \\ &+ P(X_1=-1|H_1)P(X_2=1|H_1) = \frac{1}{2}\frac{1}{2} + 2\frac{1}{4}\frac{1}{4} = \frac{3}{8}. \end{aligned}$$

Since $1/5 < 3/8$, H_1 will be chosen as the correct hypothesis.

9. **[10 points]** Let $X \sim \mathcal{N}(1,1)$. Use Chebyshev's inequality to obtain an upper bound for $P(3+|2X-2|^3 \geq 67)$.

Solution:

$$\begin{aligned} P(3+|2X-2|^3 \geq 67) &= P(|2X-2|^3 \geq 4^3) = P(|2X-2| \geq 4) \\ &= P(|X-E[X]| \geq 2) \leq \frac{\text{Var}(X)}{2^2} = \frac{1}{4}. \end{aligned}$$

10. **[4+8 points]** Let $X \sim \mathcal{N}(0,1)$ and $Y = aX + b$ for some real numbers a, b with $a > 0$. Suppose $\sigma_Y^2 = 4$.

   (a) Determine a.
   Solution: Clearly,
   $$\sigma_Y^2 = a^2\sigma_X^2 = a^2.$$
   Therefore, $a = 2$.

   (b) Assume that $Y = 0$ is observed. Find the Maximum Likelihood estimate of b for the value of a in part (a).
   Solution: Clearly, $Y \sim \mathcal{N}(a\mu_X + b, \sigma_Y^2) = \mathcal{N}(b, 4)$ (can be also computed using the scaling rule for pdfs). For b:
   $$L(b) = f_Y(0) = \frac{1}{\sqrt{8\pi}}e^{-\frac{(0-b)^2}{8}},$$
   which is maximized for $\hat{b}_{\text{ML}} = 0$.

11. **[7+7+7 points]** Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

(a) Define the *positive* random variable $Y = e^X$. Y is said to have a *lognormal* distribution with parameters μ, σ^2. Find $f_Y(y), y > 0$.
Solution:

$$F_Y(y) = P(e^X \leq y) = P(X \leq \text{ln}y) = F_X(\text{ln}y).$$

By differentiating we obtain:

$$f_Y(y) = f_X(\text{ln}y)(\text{ln}y)' = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{ln}y-\mu)^2}{2\sigma^2}}.$$

(b) Suppose that $Z = X + 2W$, where $W \sim \mathcal{N}(0, 1)$ is independent of X. Compute the unconstrained minimum MSE estimator $E[Z|X]$. Your answer should be a function of X.
Solution: X, W are jointly Gaussian since they are independent. For the same reason, Z, X are jointly Gaussian. Therefore,

$$E[Z|X] = \mu_Z + \frac{\text{Cov}(X, Z)}{\sigma_X^2}(X - \mu_X) = \mu + \frac{\sigma^2}{\sigma^2}(X - \mu) = X.$$

Alternative Solution:

$$E[Z|X] = E[X + 2W|X] = E[X|X] + 2E[W|X] = X + 2E[W] = X,$$

where the independence of X, W has been used.

(c) For Z in the part (b) compute $P(Z \geq \mu)$.
Solution: $Z \sim \mathcal{N}(\mu, \sigma^2 + 4)$. Therefore,

$$P(Z \geq \mu) = P\left(\frac{Z - \mu}{\sqrt{\sigma^2 + 4}} \geq \frac{\mu - \mu}{\sqrt{\sigma^2 + 4}}\right) = P(\tilde{Z} \geq 0) = Q(0) = \frac{1}{2},$$

where $\tilde{Z} \sim \mathcal{N}(0, 1)$.