University of Illinois Fall 2019

ECE 313: Hour Exam I

Wednesday, October 9, 2019
8:45 p.m. — 10:00 p.m.

1. **[20 points]** The two parts of this problem are unrelated.

   (a) Consider a standard deck of cards (i.e., 52 cards in total, 4 suits, and 13 cards of each suit). What is the probability of getting at least one card of each suit if you select 5 cards at random?

   Solution: In this case, I need to get a club, a diamond, a heart and a space, but the fifth card can be of any suit. There are four options to choose the repeated suit. Once the suits are fixed, I have 13 options for the card number of each of the suits that is not repeated, and $\binom{13}{2}$ for the suit that appears twice. The total number of combinations in this case is $\binom{52}{5}$, and all outcomes have the same probability as before. Hence, the sought probability is:

   $$\frac{4 \times 13 \times 13 \times 13 \times \binom{13}{2}}{\binom{52}{5}}.$$

   (b) Consider events A , B and C, with positive probability. If $P(A) = 0.5, P(A \cup C) = 0.8$ and $P(A^c B^c C^c) = 0.2$, what is $P(A^c B C^c)$? Hint: A Karnaugh map may help.

   Solution: Since $P(A \cup C) = 0.8$ and $P(A) = 0.5$, we have that $P(A^c C) = 0.3$. Hence, $P(A^c B C^c) + P(A^c B^c C^c) = 0.2$, and since $P(A^c B^c C^c) = 0.2$, we have $P(A^c B C^c) = 0$.

2. **[20 points]** You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3).

   (a) You play a game against a randomly chosen opponent. What is the probability of winning?
   Solution: Let A_i be the event of playing with an opponent of type $i \in \{1, 2, 3\}$. We have

   $$\mathbb{P}[A_1] = 0.5, \quad \mathbb{P}[A_2] = 0.25, \quad \mathbb{P}[A_3] = 0.25.$$

   Also, let WIN be the event of winning. We have

   $$\mathbb{P}[\text{WIN}|A_1] = 0.3, \quad \mathbb{P}[\text{WIN}|A_2] = 0.4, \quad \mathbb{P}[\text{WIN}|A_3] = 0.5.$$

   Thus, by the total probability theorem, the probability of winning is

   $$\begin{aligned}\mathbb{P}[\text{WIN}] =& \mathbb{P}[A_1]\mathbb{P}[\text{WIN}|A_1] + \mathbb{P}[A_2]\mathbb{P}[\text{WIN}|A_2] + \mathbb{P}[A_3]\mathbb{P}[\text{WIN}|A_3] \\ =& 0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5 \\ =& 0.375.\end{aligned}$$

   The answer is 0.375.

   (b) You play a game against a randomly chosen opponent. If you win, what is the probability of having played against an opponent of type 1?
   Solution: By the Bayes formula, we have

   $$\mathbb{P}[A_1|\text{WIN}] = \frac{\mathbb{P}[A_1]\mathbb{P}[\text{WIN}|A_1]}{\mathbb{P}[\text{WIN}]} = \frac{0.5 \times 0.3}{0.375} = \frac{2}{5}.$$

   The answer is 2/5.

3. **[20 points]** Prof. Hajek flips a fair coin repeatedly, keeping track of how many heads and how many tails he has seen, until he gets either two heads in a row or two tails in a row, at which point he stops flipping.

   (a) What is the mean number of flips until Prof. Hajek stops?

   Solution: Let L be the random variable denoting the number of flips until stopping. Prof. Hajek needs at least 2 flips, and the first 2 flips can either be in $\{HH, TT\}$, or $\{HT, TH\}$. In the first set, he would stop, whereas in the second set he would be like starting over after one flip. Hence

$$E[L] = (1/2) \cdot 2 + (1/2) \cdot (E[L] + 1).$$

   Solving this we get $E[L] = 3$.

   (b) What is the probability that Prof. Hajek gets two heads in a row (at which point he stops flipping the coin) but he also sees a second tail before he sees a second head? (e.g., THTHTHH, since the second T happens before the second H, and he stops flipping the coin with a HH).

   Solution: The sequences of flips satisfying the given condition can only be one of:

$$\{THT\mathbf{HH}, THTHT\mathbf{HH}, THTHTHT\mathbf{HH}, \ldots\}.$$

   Hence the asked probability is $(1/2)^5 + (1/2)^7 + (1/2)^9 + \ldots = (1/2)^5 \cdot \frac{1}{1-(1/2)^2} = \frac{1}{24}$.

4. **[20 points]** The two parts of this problem are unrelated.

(a) Suppose a fair die is repeatedly rolled, and let L be the number of trials conducted until the number six shows. Using the Markov inequality, compute the minimum integer, n, such that $\mathbb{P}[L \geq n] \leq 0.3$.

Solution: The random variable L has the geometric distribution with parameter $p = 1/6$. Its mean and variance is

$$\mathbb{E}[L] = \frac{1}{p}, \quad \mathrm{Var}(L) = \frac{1-p}{p^2}.$$

By applying the Markov inequality, we have

$$\mathbb{P}[L \geq n] \leq \frac{1}{n}\frac{1}{p} = \frac{6}{n}.$$

Therefore, to satisfy $\mathbb{P}[L \geq n] \leq 0.3$, n needs to be larger than or equal to 20. The answer is 20.

(b) Let X_1, X_2 be two independent discrete random variables having the same probability mass function given by:

$$p_{X_1}(x) = p_{X_2}(x) = \begin{cases} \frac{\theta}{2}, & x = 1 \\ \frac{1-\theta}{2}, & x = 2 \\ \frac{1}{2}, & x = 3 \end{cases}, \quad \text{where} \quad 0 \leq \theta \leq 1.$$

Let $Y = X_1 \cdot X_2$ and suppose that $Y = 2$ is observed. Compute the Maximum Likelihood estimate $\hat{\theta}_{ML}$ of θ.

Solution: The event $\{Y = 2\}$ occurs when $(X_1, X_2) \in \{(1,2),(2,1)\}$. Therefore,

$$L(\theta) = p_Y(2;\theta) = \frac{\theta}{2} \cdot \frac{1-\theta}{2} + \frac{1-\theta}{2} \cdot \frac{\theta}{2} = \frac{\theta(1-\theta)}{2}.$$

Differentiating with respect to θ, setting the derivative to zero and solving for θ results in $\hat{\theta}_{ML} = \frac{1}{2}$.

5. **[20 points]** Consider a binary hypothesis testing problem with the following likelihood matrix (e.g., under H_1, the probability of $X = 1$ is $1/8$):

X	1	2	3
H_1	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$
H_0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(a) Specify the ML decision rule given the observation X by breaking ties in favor of H_1. What is p_{miss}?
Solution:

X	1	2	3
H_1	$\frac{1}{8}$	$\underline{\frac{3}{8}}$	$\underline{\frac{1}{2}}$
H_0	$\underline{\frac{1}{4}}$	$\frac{1}{4}$	$\frac{1}{2}$

$$p_{miss} = P(H_0|H_1) = \frac{1}{8}.$$

(b) How many decision rules are there in which we always pick H_1 for $X = 3$?
Solution: $2^2 = 4$ decision rules.

(c) Suppose that instead of an observation of X we are given the sum of two independent realizations of X (under the same hypothesis). If the sum of these two realizations is 2, which hypothesis will the ML decision rule declare as the true hypothesis?
Solution: Sum of two independent realizations of X equal to 2 can only happen if the realized values are $(1, 1)$. This pair has probability $\frac{1}{8^2}$ under H_1 and probability $\frac{1}{4^2}$ under H_0. Thus, the ML decision rule will declare H_0 as the true hypothesis.