**University of Illinois** Fall 2019

# ECE 313: Hour Exam I

Wednesday, October 9, 2019
8:45 p.m. — 10:00 p.m.

1. **[20 points]** The two parts of this problem are unrelated.

   (a) Consider a box containing 3 Blue balls, 2 Yellow balls, 3 Red balls, and 1 Green ball. You take 3 balls at random. What is the probability that you get at least one yellow and one green balls?

   **Solution:** I could get one yellow, one green, and one of another color, or two yellow and one green (since there is only one green ball). And there are 2 yellow balls in total.Therefore, the number of options is $\binom{2}{1} \times \binom{1}{1} \times \binom{6}{1} + \binom{2}{2} \times \binom{1}{1}$. The total number of outcomes is $\binom{9}{3}$, and hence the sought probability is given by:

$$\frac{\binom{2}{1} \times \binom{1}{1} \times \binom{6}{1} + \binom{2}{2} \times \binom{1}{1}}{\binom{9}{3}}$$

   (b) Consider two events A and B with positive probability. If $A \subset B$, what is $P(B|A)$? What does $P(B)$ need to satisfy for A and B to be independent?

   **Solution:** Since $A \subset B$, $P(AB) = P(A)$, and hence $P(B|A) = 1$, and so for events A and B to be independent, we need $P(B) = 1$.

2. **[20 points]** You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3).

   (a) You play a game against a randomly chosen opponent. What is the probability of winning?
   **Solution:** Let $A_i$ be the event of playing with an opponent of type $i \in \{1, 2, 3\}$. We have

$$\mathbb{P}[A_1] = 0.5, \quad \mathbb{P}[A_2] = 0.25, \quad \mathbb{P}[A_3] = 0.25.$$

   Also, let WIN be the event of winning. We have

$$\mathbb{P}[\text{WIN}|A_1] = 0.3, \quad \mathbb{P}[\text{WIN}|A_2] = 0.4, \quad \mathbb{P}[\text{WIN}|A_3] = 0.5.$$

   Thus, by the total probability theorem, the probability of winning is

$$\begin{aligned}\mathbb{P}[\text{WIN}] =& \mathbb{P}[A_1]\mathbb{P}[\text{WIN}|A_1] + \mathbb{P}[A_2]\mathbb{P}[\text{WIN}|A_2] + \mathbb{P}[A_3]\mathbb{P}[\text{WIN}|A_3] \\ =& 0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5 \\ =& 0.375.\end{aligned}$$

   The answer is 0.375.

   (b) You play a game against a randomly chosen opponent. If you win, what is the probability of having played against an opponent of type 1?
   **Solution:** By the Bayes formula, we have

$$\mathbb{P}[A_1|\text{WIN}] = \frac{\mathbb{P}[A_1]\mathbb{P}[\text{WIN}|A_1]}{\mathbb{P}[\text{WIN}]} = \frac{0.5 \times 0.3}{0.375} = \frac{2}{5}.$$

   The answer is 2/5.

3. **[20 points]** The two parts of this problem are unrelated.

(a) Suppose a fair die is repeatedly rolled, and let $L$ be the number of trials conducted until the number six shows. Using the Chebychev inequality, compute the minimum integer, $n$, such that $\mathbb{P}[|L - \mathbb{E}[L]| \geq n] \leq 0.3$

**Solution:** The random variable $L$ has the geometric distribution with parameter $p = 1/6$. Its mean and variance are

$$\mathbb{E}[L] = \frac{1}{p}, \quad \mathrm{Var}(L) = \frac{1-p}{p^2}.$$

By applying the Chebychev inequality, we have

$$\mathbb{P}[|L - \mathbb{E}[L]| \geq n] \leq \frac{1}{n^2}\frac{1-p}{p^2} = \frac{1}{n^2}30.$$

Therefore, to satisfy $\mathbb{P}[|L - \mathbb{E}[L]| \geq n] \leq 0.3$, $n$ needs to be larger than or equal to 10. The answer is 10.

(b) Consider two identical dice, containing only numbers 1, 2, and 3. Let $X_1, X_2$ be the outcomes of rolling these dice together. Assume that the probabilities of each outcome for both $X_1$ and $X_2$ are: $p_{X_1}(1) = p_{X_2}(1) = \frac{1}{8}, p_{X_1}(2) = p_{X_2}(2) = \frac{3}{8}, p_{X_1}(3) = p_{X_2}(3) = \frac{1}{2}$. How many rolls on average are required such that $\max\{X_1, X_2\} = 2$?

**Solution:** The event $\{\max\{X_1, X_2\} = 2\}$ occurs when $(X_1, X_2) \in \{(1,2),(2,1),(2,2)\}$. Therefore, $P(\max\{X_1, X_2\} = 2) = 2 \cdot \frac{1}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{3}{8} = \frac{15}{64}$. The number of rolls $Y$ to obtain $\max\{X_1, X_2\} = 2$ is a geometric random variable with parameter $p = P(\max\{X_1, X_2\} = 2) = \frac{15}{64}$. Therefore, since $E[Y] = \frac{1}{p} = \frac{64}{15}$, $\lceil 64/15 \rceil = 5$ rolls are required on average to obtain $\max\{X_1, X_2\} = 2$.

4. **[20 points]** A robot starts at the origin and moves along the $x$-axis, one step at a time. At each step it moves forward 1 foot with probability 3/4 and backward 1 foot with probability 1/4, independently of all other steps. Let the random variable $X$ denote the position (in feet) of the robot on the $x$-axis after 5 steps.

   (a) What are the possible values of $X$?
   **Solution:** Let $Y$ be the numbers of forward steps among the 5 steps. We have $Y \sim \text{Binom}(n = 5, p = 3/4)$, and $X = Y - (5 - Y) = 2Y - 5$. Hence $X \in \{-5, -3, -1, 1, 3, 5\}$.

   (b) What is the pmf of the random variable $X$?
   **Solution:** From the pmf of binomial distribution, $P\{X = 2k - 5\} = P\{Y = k\} = \binom{5}{k}(3/4)^k(1/4)^{5-k}$, for $k = 0, 1, 2, 3, 4, 5$.

   (c) What is the expected value of $X$?
   **Solution:** $E[X] = E[2Y - 5] = 2 \cdot E[Y] - 5 = 2 \cdot 5 \cdot (3/4) - 5 = 10/4$.

5. **[20 points]** Consider a binary hypothesis testing problem with the following likelihood matrix (e.g., under $H_1$, the probability of $X = 1$ is 1/8):

| $X$ | 1 | 2 | 3 |
|---|---|---|---|
| $H_1$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{1}{2}$ |
| $H_0$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |

(a) Specify the ML decision rule given the observation $X$ by breaking ties in favor of $H_1$. What is $p_{\text{false alarm}}$?

**Solution:**

| $X$ | 1 | 2 | 3 |
|---|---|---|---|
| $H_1$ | $\frac{1}{8}$ | $\underline{\frac{3}{8}}$ | $\underline{\frac{1}{2}}$ |
| $H_0$ | $\underline{\frac{1}{4}}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |

$$p_{\text{false alarm}} = P(H_1|H_0) = \frac{3}{4}.$$

(b) How many decision rules are there?

**Solution:** $2^3 = 8$ decision rules.

(c) Suppose that instead of an observation of $X$ we are given the sum of two independent realizations of $X$ (under the same hypothesis). If the sum of these two realizations is 2, which hypothesis will the ML decision rule declare as the true hypothesis?

**Solution:** Sum of two independent realizations of $X$ equal to 2 can only happen if the realized values are $(1, 1)$. This pair has probability $\frac{1}{8^2}$ under $H_1$ and probability $\frac{1}{4^2}$ under $H_0$. Thus, the ML decision rule will declare $H_0$ as the true hypothesis.