University of Illinois Fall 2019

ECE 313: Hour Exam II

Wednesday, November 13, 2019
8:45 p.m. — 10:00 p.m.

1. **[20 (7+8+5) points]** Consider a continuous random variable X with the following pdf:

$$f_X(u) = \begin{cases} K\lambda e^{-\lambda u}, & 0 \le u \le a \\ \\ 0, & \text{otherwise} \end{cases},$$

for some positive constants K, a, and λ.

(a) What must K, a, and λ satisfy for f_X to be a valid pdf?
Solution: For f_X to be a valid pdf, we must have:

$$\begin{aligned} 1 &= \int_0^a K\lambda e^{-\lambda u} du \\ &= -Ke^{-\lambda u}|_0^a \\ &= K - Ke^{-\lambda a} \\ &= K(1 - e^{-\lambda a}) \end{aligned}$$

Alternatively, one can directly use $\int_0^a \lambda e^{-\lambda u} du = F_Y(a) = 1 - e^{-\lambda a}$, with $Y \sim Exp(\lambda)$. Therefore, we have

$$a = -\frac{1}{\lambda}\ln(1 - 1/K)$$

For $K = 1$, $a = \infty$, and $X \sim Exp(\lambda)$. For $K = 2$, $a = -\frac{1}{\lambda}\ln(1/2)$.

(b) What is the probability of $X > a/2$ if $X > a/4$? In other words, compute $P\{X > \frac{a}{2}|X > \frac{a}{4}\}$. Leave your answer in terms of K and a.

Solution: First note that $F_X(c) = 0$ for $c < 0$, $F_X(c) = 1$ for $c > a$, and for $0 \le c \le a$:

$$\begin{aligned} F_X(c) &= \int_0^c K\lambda e^{-\lambda u} du \\ &= K(1 - e^{-\lambda c}) \end{aligned}$$

$$\begin{aligned} P\{X > \frac{a}{2}|X > \frac{a}{4}\} &= \frac{P\{X > \frac{a}{2}, X > \frac{a}{4}\}}{P\{X > \frac{a}{4}\}} \\ &= \frac{P\{X > \frac{a}{2}\}}{P\{X > \frac{a}{4}\}} \\ &= \frac{1 - F_X(\frac{a}{2})}{1 - F_X(\frac{a}{4})} \\ &= \frac{1 - K(1 - e^{-\lambda a/2})}{1 - K(1 - e^{-\lambda a/4})} \end{aligned}$$

Alternatively,

$$\begin{aligned} \frac{P\{X > \frac{a}{2}\}}{P\{X > \frac{a}{4}\}} &= \frac{\int_{a/2}^a K\lambda e^{-\lambda u} du}{\int_{a/4}^a K\lambda e^{-\lambda u} du} \\ &= \frac{e^{-\lambda a/2} - e^{-\lambda a}}{e^{-\lambda a/4} - e^{-\lambda a}} \end{aligned}$$

(c) Compute $P\{X = \frac{a}{2}\}$ and $P\{X^2 + aX > 0\}$. Leave your answer in terms of K and a.

Solution: $P\{X = \frac{a}{2}\} = 0$, since X is a continuous random variable. $P\{X^2 + aX > 0\} = 1$, since $X > 0$ and $a > 0$.

2. **[20 (7 + 13) points]** Corey is doing research on traffic flow in which he assumes that cars pass by the Grainger library crossing on Springfield Ave during the lunch hours according to a Poisson process with rate λ cars per minute. Based on that assumption, solve the following.

   (a) Find the average waiting time to see the next passing car? Justify your answer.
   Solution: The waiting time U between passing cars has exponential distribution: $U \sim$ Exponential(λ). Hence $E[U] = 1/\lambda$ (minute).

(b) Find the probability that there are 3 passing cars between 12:01pm and 12:03pm given that there are 2 passing cars between 12:00pm and 12:02pm.

Solution: Let A be the event that there are 2 passing cars between 12:00pm and 12:02pm, and B be the event that there are 3 passing cars between 12:01pm and 12:03pm. Note that these two intervals are not disjoint, so we need to break these them into disjoint intervals to get independent Poisson random variables. Let $E_{i,j,k}$ be the event that there are:

i. i passing cars from 12:00pm to 12:01pm,

ii. j passing cars from 12:01pm to 12:02pm,

iii. k passing cars from 12:02pm to 12:03pm.

Then $P(AB)$ is the sum of probabilities of the following of disjoint events as

$$P(AB) = P(E_{2,0,3}) + P(E_{1,1,2}) + P(E_{0,2,1}).$$

Using the Poisson process assumption, we have

$$P(E_{i,j,k}) = \frac{e^{-\lambda}\lambda^i}{i!}\frac{e^{-\lambda}\lambda^j}{j!}\frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-3\lambda}\lambda^{i+j+k}}{i!\,j!\,k!}.$$

Hence,

$$P(AB) = e^{-3\lambda}\left(\frac{\lambda^5}{2!\,0!\,3!} + \frac{\lambda^4}{1!\,1!\,2!} + \frac{\lambda^3}{0!\,2!\,1!}\right).$$

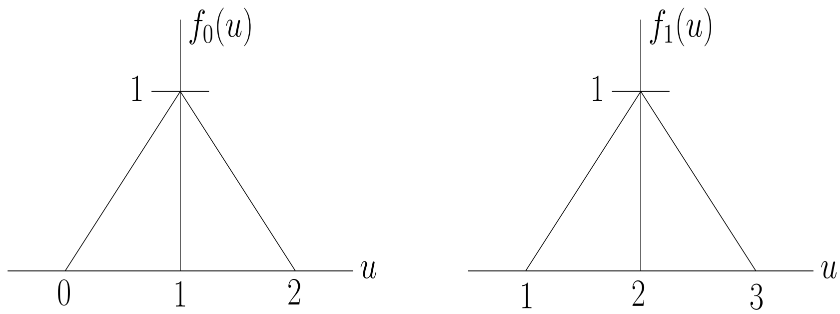
Finally, the asked conditional probability is

$$\begin{aligned} P(B|A) &= \frac{P(AB)}{P(A)} \\ &= e^{-3\lambda}\left(\frac{\lambda^5}{2!\,0!\,3!} + \frac{\lambda^4}{1!\,1!\,2!} + \frac{\lambda^3}{0!\,2!\,1!}\right)\frac{2!}{e^{-2\lambda}\,(2\lambda)^2} \\ &= e^{-\lambda}\left(\frac{\lambda^3}{24} + \frac{\lambda^2}{4} + \frac{\lambda}{4}\right). \end{aligned}$$

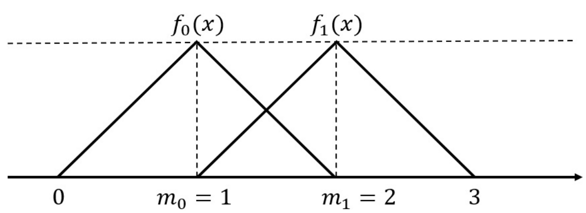
3. **[10 points]** Let $X \sim N(3, 16)$. Clearly describe how can we use the Q table to find $P(1 \leq X < 5)$. **Solution:** Again using standardized version of X, we have

$$\begin{aligned}
P(1 \leq X < 5) &= P\left(\frac{1-3}{4} \leq \frac{X-3}{4} < \frac{5-3}{4}\right) \\
&= P(-0.5 \leq Z < 0.5) \\
&= 1 - P(Z < -0.5) - P(Z \geq 0.5) \\
&= 1 - 2 \cdot Q(0.5).
\end{aligned}$$

4. **[10 points]** Assume that if hypothesis 0 (H_0) is true, then the random variable X has the pdf $f_0(x)$, and if hypothesis 1 (H_1) is true, then the random variable X has the pdf $f_1(x)$, whose graphs are given in the following figure. Find the ML decision rule for the hypothesis test, and the corresponding p_{miss}.



Solution: Intersection of graphs of f_0 and f_1 is $x = 3/2$ (see the picture below).



Therefore, if we observe $X = x$, then the ML rule is

$$\begin{cases} f_1(x) \geq f_0(x) & \text{if } x \geq 3/2 \\ f_1(x) < f_0(x) & \text{if } x < 3/2 \end{cases}$$

Therefore, the ML decision rule is

$$\begin{cases} \text{Declare } H_1 & \text{if } x \geq 3/2 \\ \text{Declare } H_0 & \text{if } x < 3/2 \end{cases}$$

We have $p_{\text{miss}} = \mathbb{P}[\text{Declar } H_0 \text{ true} | H_1] = \int_{-\infty}^{3/2} f_1(x) dx = \frac{1}{8}$.

5. **[20 points]** Prof. Hajek is driving from Champaign to Chicago at a constant speed X miles per hour, which is a random variable uniformly distributed in the interval $[70, 80]$. The distance between Champaign and Chicago is 140 miles. Let random variable Y be the time duration (in hours) of the trip, i.e., $Y = g(X) = \frac{140}{X}$. Find the pdf of Y.

   Solution: We first find the CDF, and then we derivate it to find the pdf. First note that since $X \in [70, 80]$, $Y \in [14/8, 2]$. Therefore, $F_Y(y) = 0$ for $y < 14/8$, and $F_Y(y) = 1$ for $y > 2$. For $14/8 \leq y \leq 2$, we have

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[140/y \leq X] = (80 - \frac{140}{y})\frac{1}{10} = 8 - \frac{14}{y},$$

   where we have used the fact that $X \sim U[70, 80]$.

   Finally, taking the derivative, we get $f_Y(y) = \frac{14}{y^2}$, for $y \in [14/8, 2]$, and 0 otherwise.

6. **[20 points]** Let X, Y be two independent exponential random variables with $\lambda = 1$. Consider the square $\mathcal{S}$ with corner points $(0,0), (0,r), (r,0), (r,r)$. Find r such that $P((X,Y) \in \mathcal{S}) = \left(1 - e^{-2}\right)^2$.

Solution:

$$\begin{aligned} P((X,Y) \in \mathcal{S}) &= P(0 \le X \le r, 0 \le Y \le r) = P(0 \le X \le r)P(0 \le Y \le r) \\ &= F_X(r)F_Y(r) = (1 - e^{-r})^2. \end{aligned}$$

Here, the independence and the nonnegativity of X, Y have been employed. By matching the expression with the desired probability $\left(1 - e^{-2}\right)^2$ we conclude that

$$r = 2.$$