University of Illinois Fall 2019

ECE 313: Hour Exam II

Wednesday, November 13, 2019
8:45 p.m. — 10:00 p.m.

1. **[20 (7 + 7 + 6) points]** Consider a continuous random variable X with the following pdf:

$$f_X(u) = \begin{cases} K(1-u^2), & -a \le u \le a \\ \\ 0, & \text{otherwise} \end{cases},$$

for some constants $K > 0$ and $a > 0$.

(a) What must K and a satisfy for f_X to be a valid pdf? Can a be greater than one?
Solution: For f_X to be a valid pdf, we must have:

$$\begin{aligned} 1 &= \int_{-a}^{a} K(1-u^2)du \\ &= K(u - \frac{u^3}{3})|_{-a}^{a} \\ &= K(a - \frac{a^3}{3} + a - \frac{a^3}{3}) \\ &= K(2a - \frac{2a^3}{3}) \end{aligned}$$

Therefore, for $a = 1$, $K = 3/4$.
Finally, a can not be greater than 1, since the pdf would be negative in that case.

(b) Compute the CDF of X, i.e., F_X. Leave your answer in terms of K and a.
Solution: Note that $F_X(c) = 0$ for $c < -a$, $F_X(c) = 1$ for $c > a$, and for $-a \le c \le a$:

$$\begin{aligned} F_X(c) &= \int_{-a}^{c} K(1-u^2)du \\ &= K(u - \frac{u^3}{3})|_{-a}^{c} \\ &= K(c - \frac{c^3}{3} + a - \frac{a^3}{3}) \end{aligned}$$

(c) For $a = 1$, compute $P(X^2 + 2X > 0)$ and $E[X]$.

Solution:

$$\begin{aligned} P(X^2 + 2X > 0) &= P(X(X+2) > 0) \\ &= \int_0^1 K(1-u^2)du \\ &= K(u - \frac{u^3}{3})|_0^1 \\ &= K\frac{2}{3} = 1 \end{aligned}$$

One can also realize that in this case, $\int_0^1 K(1-u^2)du = 1/2$, since the pdf is symmetric and we are integrating over the right half.

$E[X] = 0$, since the pdf is symmetric around the origin.

2. **[20 (7 + 13) points]** Corey is doing research on traffic flow in which he assumes that cars pass by the Grainger library crossing on Springfield Ave during the lunch hours according to a Poisson process with rate λ cars per minute. Based on that assumption, solve the following.

   (a) Find the probability that there are 2 passing cars between 12:00pm and 12:02pm.
   Solution: Let N be the number of passing cars between 12:00pm and 12:02pm. Then according to Poisson process assumption, $N \sim \text{Poisson}(2\lambda)$. Thus

$$P(N = 2) = \frac{e^{-2\lambda}\,(2\lambda)^2}{2!}$$

(b) Find the probability that there are 3 passing cars between 12:01pm and 12:03pm given that there are 2 passing cars between 12:00pm and 12:02pm.

Solution: Let A be the event that there are 2 passing cars between 12:00pm and 12:02pm, and B be the event that there are 3 passing cars between 12:01pm and 12:03pm. Note that these two intervals are not disjoint, so we need to break these them into disjoint intervals to get independent Poisson random variables. Let $E_{i,j,k}$ be the event that there are:

i. i passing cars from 12:00pm to 12:01pm,
ii. j passing cars from 12:01pm to 12:02pm,
iii. k passing cars from 12:02pm to 12:03pm.

Then $P(AB)$ is the sum of probabilities of the following of disjoint events as

$$P(AB) = P(E_{2,0,3}) + P(E_{1,1,2}) + P(E_{0,2,1}).$$

Using the Poisson process assumption, we have

$$P(E_{i,j,k}) = \frac{e^{-\lambda}\lambda^i}{i!} \frac{e^{-\lambda}\lambda^j}{j!} \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-3\lambda}\lambda^{i+j+k}}{i!\, j!\, k!}.$$

Hence,

$$P(AB) = e^{-3\lambda}\left(\frac{\lambda^5}{2!\,0!\,3!} + \frac{\lambda^4}{1!\,1!\,2!} + \frac{\lambda^3}{0!\,2!\,1!}\right).$$

Finally, the asked conditional probability is

$$\begin{aligned} P(B|A) &= \frac{P(AB)}{P(A)} \\ &= e^{-3\lambda}\left(\frac{\lambda^5}{2!\,0!\,3!} + \frac{\lambda^4}{1!\,1!\,2!} + \frac{\lambda^3}{0!\,2!\,1!}\right) \frac{2!}{e^{-2\lambda}\,(2\lambda)^2} \\ &= e^{-\lambda}\left(\frac{\lambda^3}{24} + \frac{\lambda^2}{4} + \frac{\lambda}{4}\right). \end{aligned}$$

3. **[10 points]** Let $X \sim N(3, 16)$. Clearly describe how can we use the Q table to find u such that $P(X > u) = 0.05$.

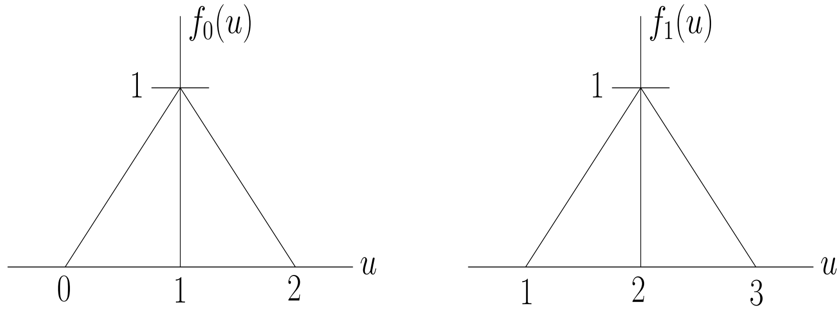
Solution: We have

$$P(X > u) = P\left(\frac{X-3}{4} > \frac{u-3}{4}\right) = P\left(Z > \frac{u-3}{4}\right) = Q\left(\frac{u-3}{4}\right).$$

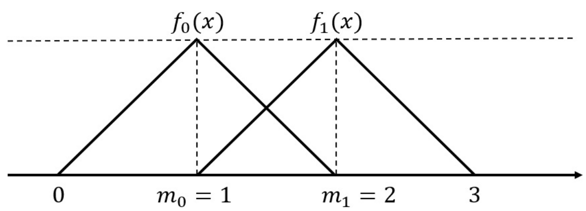
Hence using the Q table we can find

$$\frac{u-3}{4} = Q^{-1}(0.05) \implies u = 4 \cdot Q^{-1}(0.05) + 3.$$

4. **[10 points]** Assume that if hypothesis 0 (H_0) is true, then the random variable X has the pdf f_0, and if hypothesis 1 (H_1) is true, then the random variable X has the pdf f_1, whose graphs are given in the following figure. Find the ML decision rule for the hypothesis test, and the corresponding $p_{\text{false alarm}}$.



Solution: Intersection of graphs of f_0 and f_1 is $x = 3/2$ (see the picture below).



Therefore, if we observe $X = x$, then the ML rule is

$$\begin{cases} f_1(x) \geq f_0(x) & \text{if } x \geq 3/2 \\ f_1(x) < f_0(x) & \text{if } x < 3/2 \end{cases}$$

Therefore, the ML decision rule is

$$\begin{cases} \text{Declare } H_1 & \text{if } x \geq 3/2 \\ \text{Declare } H_0 & \text{if } x < 3/2 \end{cases}$$

We have $p_{\text{false alarm}} = \mathbb{P}[\text{Declar } H_1 \text{ true}|H_0] = \int_{3/2}^{\infty} f_0(x)dx = \frac{1}{8}$.

5. **[20 points]** Prof. Hajek is driving from Champaign to Chicago at a constant speed X miles per hour, which is a random variable uniformly distributed in the interval $[70, 80]$. The distance between Champaign and Chicago is 140 miles. Let random variable Y be the time duration (in hours) of the trip, i.e., $Y = g(X) = \frac{140}{X}$. Find the pdf of Y.

   Solution: We first find the CDF, and then we derivate it to find the pdf. First note that since $X \in [70, 80]$, $Y \in [14/8, 2]$. Therefore, $F_Y(y) = 0$ for $y < 14/8$, and $F_Y(y) = 1$ for $y > 2$. For $14/8 \leq y \leq 2$, we have

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[140/y \leq X] = (80 - \frac{140}{y})\frac{1}{10} = 8 - \frac{14}{y},$$

   where we have used the fact that $X \sim U[70, 80]$.

   Finally, taking the derivative, we get $f_Y(y) = \frac{14}{y^2}$, for $y \in [14/8, 2]$, and 0 otherwise.

6. **[20 (10+10) points]** Consider the joint density function:

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}.$$

(a) Compute $P(X < Y)$.
Solution:

$$P(X<Y) = \iint_{(x,y):x<y} f_{XY}(x,y)dxdy = \int_0^\infty \int_0^y f_{XY}(x,y)dxdy = \int_0^\infty \int_0^y 6e^{-(2x+3y)}dxdy$$
$$\int_0^\infty 3e^{-3y} \int_0^y 2e^{-2x}dxdy = \int_0^\infty 3e^{-3y}\left[1-e^{-2y}\right]dy = \int_0^\infty 3e^{-3y}dy - \int_0^\infty 3e^{-5y}dy$$
$$= 1 - 3\left[-\frac{e^{-5y}}{5}\right]_0^\infty = 1 - \frac{3}{5} = \frac{2}{5}.$$

Here, we have used the observation that $2e^{-2x}$ is the pdf of an Exp(2) random variable X. Denote by $F_X(x)$ the corresponding CDF. Then, $\int_0^y 2e^{-2x}dx = F_X(y) = 1 - e^{-2y}$. Similarly, $3e^{-3y}$ is the pdf of an Exp(3) random variable Y and therefore, $\int_0^\infty 3e^{-3y}dy = 1$.

(b) Are X, Y independent?
Solution: Clearly, $f_{XY}(x,y) = f_X(x)f_Y(y)$ for $0 < x < \infty, 0 < y < \infty$, where $f_X(x) = 2e^{-2x}, x > 0$ and $f_Y(y) = 3e^{-3y}, y > 0$. Therefore, X, Y are independent.