University of Illinois Fall 2018

ECE 313: Final Exam

Tuesday, December 18, 2018
1:30 p.m. — 4:30 p.m.

1. **[6+6+6 points]** Consider events A, B, C and D with probabilities $P(A) = 1/3$, $P(B) = 3/5$, $P(C) = 2/5$, and $P(D) = 3/5$, and suppose that $P(B|A) = 1/2$.

   (a) Find $P(A^cB)$.
   Solution:
   $$P(A^cB) = P(B) - P(AB) = P(B) - P(B|A)P(A) = \frac{3}{5} - \frac{1}{2} \times \frac{1}{3} = \frac{13}{30}$$

   (b) Find $P(A|B)$.
   Solution:
   $$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{1/2 \times 1/3}{3/5} = \frac{5}{18}$$

   (c) If A and C are independent, find $P(A^cC)$,
   Solution:
   $$P(A^cC) = P(A^c)P(C) = \left(1 - \frac{1}{3}\right)\frac{2}{5} = \frac{4}{15}$$

2. **[10+6 points]** Two sensors are used to detect whether a patient has sepsis. The first sensor outputs a value X and the second sensor outputs a value Y. Both outputs have possible values $0, 1, 2$, with larger numbers tending to indicate that the patient has sepsis. Suppose

	$X = 0$	$X = 1$	$X = 2$
H_1	0.1	0.3	0.6
H_0	0.6	0.2	0.2

	$Y = 0$	$Y = 1$	$Y = 2$
H_1	0.1	0.1	0.8
H_0	0.7	0.2	0.1

   given one of the hypotheses is true, the sensors provide conditionally independent readings, i.e., $P(X, Y|H_i) = P(X|H_i)P(Y|H_i)$ for $i = 0, 1$.

   (a) Suppose, based on past experience, prior probabilities $\pi_1 = 0.2$ and $\pi_0 = 0.8$ are assigned. Compute the joint probability matrix and indicate the MAP decision rule.
   Solution: The joint probability matrix is given by The MAP decisions are indicated

(X,Y)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
H_1	0.002	0.002	0.016	0.006	0.006	$\underline{0.048}$	0.012	0.012	$\underline{0.096}$
H_0	$\underline{0.0336}$	$\underline{0.096}$	$\underline{0.048}$	$\underline{0.112}$	$\underline{0.032}$	0.016	$\underline{0.112}$	$\underline{0.032}$	0.016

   by the underlined elements in the joint probability matrix. The larger number in each column is underlined

   (b) For the MAP decision rule found in part (a), compute $p_{\text{false alarm}}$, p_{miss}, and the probability of error p_e.
   Solution: For the MAP rule, $p_{\text{false alarm}} = P((X, Y) \in \{(1, 2), (2, 2)\}|H_0) = 0.02+0.02 = 0.04$, and $p_{\text{miss}} = 1 - P((X, Y) \in \{(1, 2), (2, 2)\}|H_1) = 1 - 0.24 - 0.48 = 0.28$. Thus, for the MAP rule, $p_e = 0.8 \times 0.04 + 0.2 \times 0.28 = 0.088$.

3. **[8+6+8 points]** The three parts are unrelated.

(a) Suppose X is a binomial random variable with parameters $n = 16$ and $p = 1/2$. Using the Central Limit Theorem, express $P(X \geq 10)$ in terms of the Q function **without** using the continuity correction.

Solution: We note that $E[X] = np = 8$ and $\text{Var}(X) = np(1-p) = 16(1/2)(1/2) = 4$. Using the CLT, we approximate X by $\tilde{X} \sim \mathcal{N}(E[X], \text{Var}(X))$. Therefore, we have:

$$P(X \geq 10) \approx P(\tilde{X} \geq 10) = P\left(\frac{\tilde{X}-8}{\sqrt{4}} \geq \frac{10-8}{\sqrt{4}}\right) = Q(1).$$

(b) Assume that people show up from the corner of a near building to your place according to a Poisson process with rate $\lambda = 2$ people per hour. Find the probability of at least 3 people showing up in the next 2 hours. You can leave your answer in terms of e, the base of natural logarithm, e.g. $2e^{-1}$.

Solution:

$$\begin{aligned} P(N(2) \geq 3) &= 1 - P(N(2) = 0) - P(N(2) = 1) - P(N(2) = 2) \\ &= 1 - \sum_{k=0}^{2} e^{-4} \frac{4^k}{k!} = 1 - 13e^{-4}. \end{aligned}$$

(c) Suppose that in your kitchen there is a box with n apples. You particularly like apples, therefore every day you remove an apple from the box and you eat it. To avoid a fruit shortage in your home, your mother replaces every day the fruit that you ate by an apple with probability p or by an orange with probability $1-p$. Find the expected number of days till there are no more apples in the box.

Solution: Each day, an apple is totally removed from the box with probability $1-p$ and the number of apples decreases by 1. Also, if at a particular day the box contains k apples, the box will contain at most k apples in any subsequent day, since you definitely eat an apple every day. The number of days required to finish the apples in the box is a negative binomial random variable with parameters n and $1-p$. Therefore, the expected number of days to eat all apples is $n/(1-p)$.

4. **[8+8+4 points]** Suppose R has a Rayleigh pdf given by:

$$f_R(u) = \begin{cases} 2u\, e^{-u^2} & \text{if } u \geq 0 \\ 0 & \text{else.} \end{cases}$$

Let $X = R^2$.

(a) Find $P\{R > 5 \mid R > 2\}$.

Solution: Note that for $c > 0$,

$$P\{R > c\} = \int_c^{\infty} 2u\, e^{-u^2} du = \int_{c^2}^{\infty} e^{-t} dt = e^{-c^2}$$

Thus

$$P\{R > 5 \mid R > 2\} = \frac{P\{R > 5, R > 2\}}{P\{R > 2\}} = \frac{P\{R > 5\}}{P\{R > 2\}} = \frac{e^{-25}}{e^{-4}} = e^{-21}.$$

(b) Find the pdf of X.
Solution: We first compute the CDF of X. Clearly $F_X(c) = 0$ for $c < 0$. For $c \geq 0$,

$$F_X(c) = P\{R^2 \leq c\} = P\{R \leq \sqrt{c}\} = \int_0^{\sqrt{c}} 2u\, e^{-u^2} du = \int_0^c e^{-t} dt = 1 - e^{-c}$$

Thus, $f_X(c) = 0$ for $c < 0$, and for $c \geq 0$,

$$f_X(c) = e^{-c}$$

which means that X is an Exp(1) random variable.

(c) Find $P\{X > 5 \mid X > 2\}$.
Solution: Since X has a memoryless distribution,

$$P\{X > 5 \mid X > 2\} = P\{X > 3\} = e^{-3}.$$

But we can also conclude this by computing the expression using the pdf of R.

5. **[8+6 points]** Consider a 6×6 square board, which consists of 36 squares in 6 rows and 6 columns.

(a) How many different rectangles, comprised entirely of the board squares, can be drawn on the board? *Hint:* there are 7 horizontal and 7 verticle lines in the board.
Solution: A rectangle is uniquely described by the pair of horizontal lines and the pair of vertical lines that form its sides. Since there are $\binom{7}{2} = \frac{7 \times 6}{2} = 21$ choices for the pair of horizontal lines, an, similarly, 21 choices for the pair of vertical lines, there are $21 \times 21 = 441$ rectangles

(b) One of the rectangles you counted in part (a) is chosen at random. What is the probability that it is a square?
Solution: The number of square shaped rectangles is $(7-k)^2$, Hence, the number of square shaped rectangles is $1^2 + 2^2 + 3^2 + \cdots + 6^2 = 7 \times 13$. So the probability of getting a square shaped rectangle is $\frac{13}{63}$.

6. **[8+8+8 points]** Suppose X and Y are independent random variables with X being Exp(1) and Y being Exp(3), i.e.,

$$f_{X,Y}(u,v) = \begin{cases} 3e^{-u}e^{-3v} & \text{if } u \geq 0, v \geq 0 \\ 0 & \text{else.} \end{cases}$$

(a) Find $P\{X > Y\}$.
Solution:

$$\begin{aligned} P\{X > Y\} &= \int_{v=0}^{\infty} \left(\int_{u=v}^{\infty} e^{-u} du \right) 3e^{-3v} dv \\ &= \int_0^{\infty} 3e^{-v}e^{-3v} dv = -\frac{3}{4}\, e^{-4v}\Big|_0^{\infty} = \frac{3}{4}. \end{aligned}$$

(b) Find the pdf of $W = \min\{X, Y\}$.
Solution: We first find the CDF of W. Clearly $F_W(c) = 0$, for $c < 0$. For $c \geq 0$,

$$\begin{aligned} F_W(c) &= 1 - P\{\min\{X,Y\} > c\} = 1 - P\{X > c, Y > c\} \\ &= 1 - P\{X > c\}P\{Y > c\} = 1 - e^{-c}e^{-3c} = 1 - e^{-4c}. \end{aligned}$$

Thus $f_W(c) = 0$, for $c < 0$, and for $c \geq 0$

$$f_W(c) = 4e^{-4c}$$

i.e., W is an Exp (4) random variable.

(c) Find the pdf of $S = X + Y$.
Solution: Since X and Y are independent, we can apply the convolution formula to compute the pdf of S. Clearly $f_S(c) = 0$, for $c < 0$. For $c \geq 0$,

$$\begin{aligned} f_S(c) &= \int_{-\infty}^{\infty} f_X(u) f_Y(c-u) du = \int_0^c 3e^{-u}\, e^{-3(c-u)} du \\ &= 3e^{-3c} \int_0^c e^{2u} du = \frac{3}{2} e^{-3c}(e^{2c} - 1) = \frac{3}{2}(e^{-c} - e^{-3c}). \end{aligned}$$

7. **[8+6 points]** Let X and Y be independent random variables, both with mean 0 and variance 1. Define the random variables

$$V = 2X + 3Y \quad \text{and} \quad W = X - Y.$$

(a) Compute the the linear MMSE estimator $\hat{E}[V|W]$.
Solution:

$$\hat{E}[V|W] = E[V] + \frac{\text{Cov}(V, W)}{\text{Var}(W)}(W - E[W]) = \frac{\text{Cov}(V, W)}{\text{Var}(W)} W.$$

We now compute $\text{Cov}(V, W) = E[(2X + 3Y)(X - Y)] = 2\text{Var}(X) - 3\text{Var}(Y) = -1$ and $\text{Var}(W) = \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 2$. Therefore,

$$\hat{E}[V|W] = -\frac{1}{2} W.$$

(b) Assume instead that W is defined as $W = X - aY$ for some real a. Can V and W be uncorrelated for some value of a? Justify your answer.
Solution: Setting $\text{Cov}(V, W) = 0$, we obtain:

$$0 = \text{Cov}(V, W) = E[(2X + 3Y)(X - aY)] = 2E[X^2] - 3aE[Y^2] = 2 - 3a.$$

Therefore, V, W are uncorrelated for $a = 2/3$.

8. **[8 points]** Suppose X_1, X_2, ... X_n is a sequence of random variables such that each X_k has finite mean μ and variance 2, and $\text{Cov}(X_i, X_j) = -\frac{1}{n}$ for $i \neq j$. Let $S_n = \sum_{k=1}^n X_k$. For a given $\delta > 0$, use Chebychev inequality to obtain an upper bound of

$$P\left\{\left|\frac{S_n}{n} - \mu\right| \geq \delta\right\}.$$

Solution: The mean of $\frac{S_n}{n}$ is given by

$$E\left[\frac{S_n}{n}\right] = E\left[\frac{\sum_{k=1}^n X_k}{n}\right] = \frac{\sum_{k=1}^n E[X_k]}{n} = \frac{n\mu}{n} = \mu.$$

The variance of $\frac{S_n}{n}$ is given by:

$$\begin{aligned} \text{Var}\left(\frac{S_n}{n}\right) &= \text{Var}\left(\frac{\sum_{k=1}^n X_k}{n}\right) = \frac{\text{Cov}\left(\sum_{k=1}^n X_k, \sum_{k=1}^n X_k\right)}{n^2} \\ &= \frac{\sum_{k=1}^n \text{Var}(X_k) + \sum_{i\neq j} \text{Cov}(X_i, X_j)}{n^2} = \frac{2n + n(n-1)(-\frac{1}{n})}{n^2} = \frac{n+1}{n^2} \end{aligned}$$

Using Chebyshev,

$$P\left\{\left|\frac{S_n}{n} - \mu\right| \geq \delta\right\} \leq \frac{\text{Var}\left(\frac{S_n}{n}\right)}{\delta^2} = \frac{n+1}{n^2\delta^2}.$$

9. **[6+8+6 points]** Let X and Y be jointly Gaussian random variables with $\mu_X = 0$, $\mu_Y = 1$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 1$.

   (a) If $\rho = \frac{1}{8}$, find $P(X + 2Y > 2)$.
   Solution: Since $X + 2Y$ is a linear combination of jointly Gaussian random variables, it is a Gaussian random variable. $E(X + 2Y) = \mu_X + 2\mu_Y = 2$. Since a Gaussian random variable is symmetric with respect to its mean, $P(X + 2Y > 2) = P(X + 2Y > E(X + 2Y)) = 0.5$.

   (b) If $\rho = \frac{1}{2}$, find $E[Y|X]$.
   Solution: Since X and Y are jointly Gaussian random variables,

   $$E[Y|X] = \hat{E}[Y|X] = \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(X - \mu_X) = 1 + \frac{X}{4}.$$

   (c) If $\rho = 0$, find $f_{Y|X}(v|u)$.
   Solution: Since X and Y are jointly Gaussian random variables and $\rho = 0$, X and Y are independent. Hence

   $$f_{Y|X}(v|u) = f_Y(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(v-1)^2}{2}},$$

   since Y is a Gaussian random variable with mean μ_Y and variance σ_Y^2.

10. **[8+6 points]** The two parts are unrelated.

   (a) A random observation X is sampled from a Poisson distribution with parameter λ. Suppose that you toss X times a biased coin with $P(\text{Heads}) = p$. Compute the probability mass function $P(Y = k)$ for any integer $k \geq 0$, where Y is the number of heads that occur in this experiment. **Hint**: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
   Solution: The law of total probability gives:

   $$\begin{aligned} P(Y = k) &= \sum_{n=k}^{\infty} P(Y = k|X = n)P(X = n) = \sum_{n=k}^{\infty} \binom{n}{k} p^k(1-p)^{n-k} \frac{e^{-\lambda}}{n!}\lambda^n \\ &= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k(1-p)^{n-k} \frac{e^{-\lambda}}{n!}\lambda^n = \frac{e^{-\lambda}}{k!}\lambda^k p^k \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} \\ &= \frac{e^{-\lambda}}{k!}\lambda^k p^k \sum_{r=0}^{\infty} \frac{[\lambda(1-p)]^r}{r!} = \frac{e^{-\lambda}}{k!}\lambda^k p^k e^{\lambda(1-p)} = \frac{e^{-\lambda p}}{k!}(\lambda p)^k, \quad k = 0, 1, 2, \ldots. \end{aligned}$$

   Therefore Y is a Poisson random variable with mean value λp.

(b) A box contains 3 white and 6 black balls. Balls are randomly selected, one at a time, until a white one is obtained. If we assume that each selected ball is replaced by a ball of the same color before the next one is drawn, what is the probability that at least 3 draws are required?

Solution: Let X be the number of draws until a white ball is selected. Due to drawing balls with replacement, $X \sim \text{Geo}(p)$ with probability of success

$$p = \frac{3}{3+6} = \frac{1}{3}.$$

Therefore,

$$P(X \geq 3) = P(X > 2) = (1-p)^2.$$

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Consider the events such that $P(ABC) = P(B)P(AC) > 0$ and $P(BC) = P(B)P(C)$.

TRUE	FALSE	
☐	☐	$P(A\|BC) = P(A\|C)$.
☐	☐	$P(B\|AC) = P(B)$.
☐	☐	If $P(A) < P(C)$, then $P(A\|C) > P(C\|A)$.

Solution: True, True, False

(b) Suppose a coin shows head with unknown probability p. Three experiments are conducted. In the first experiment, the coin is flipped 10 times and the number of heads is denoted by X. In the second experiment, the coin is flipped another 10 times and the number of heads is denoted by Y. In the third experiment, the coin is flipped another 20 times and the number of heads is denoted by Z.

TRUE	FALSE	
☐	☐	Given $X = 2$, the ML estimate of p is 0.2.
☐	☐	Given $X = 2$ and $Y = 4$, the ML estimate of p is $\frac{0.2+0.4}{2} = 0.3$.
☐	☐	Given $X = 2$ and $Z = 5$, the ML estimate of p is $\frac{0.2+0.25}{2} = 0.225$.

Solution: True,True, False,

(c) Let $X \sim \mathcal{N}(0,1)$ and $I \sim \text{Ber}(1/2)$ be independent random variables. Define the random variable Y as follows:

$$Y = \begin{cases} X, & \text{if } I = 1 \\ -X, & \text{if } I = 0 \end{cases}.$$

TRUE	FALSE	
☐	☐	X, Y are independent random variables.
☐	☐	$Y \sim \mathcal{N}(0,1)$.

Solution: False,True

(d) Suppose U_1, U_2, ... U_n is a sequence of i.i.d. random variables such that each U_k has a uniform distribution over $[0, c]$. Consider the product $\prod_{k=1}^{n} U_k$ as $n \to \infty$.

☐ ☐ If $c = 3$, $P(\prod_{k=1}^{n} U_k > \delta) \to 0$ as $n \to \infty$ for any $\delta > 0$.

☐ ☐ If $c = 4$, $P(\prod_{k=1}^{n} U_k > C) \to 1$ as $n \to \infty$ for any $C > 0$.

Solution: False, True