University of Illinois Fall 2018

ECE 313: Hour Exam II

Wednesday, November 14, 2018
8:45 p.m. — 10:00 p.m.

1. **[8+6+10+6 points]** The joint pmf of two discrete-type random variables is as shown in the table below

	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	0.2	0	0.1	0.1
$Y=2$	0	0.25	0.15	0.05
$Y=3$	0.05	0	0	0.1

(a) Find the marginal pmfs of X and Y.
Solution: The marginal pmf of X is given by the column sums. Thus

$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = 0.25.$$

The marginal pmf of Y is given by the row sums. Thus

$$p_Y(1) = 0.4, \quad p_Y(2) = 0.45, \quad p_Y(3) = 0.15.$$

(b) Are X and Y independent? Justify your answer.
Solution: No. For example, $p_{X,Y}(1,1) = 0.2$ and $p_X(1)p_Y(1) = 0.25 \times 0.4 = 0.1$. Thus $p_{X,Y}(1,1) \neq p_X(1)p_Y(1)$.

(c) Find the conditional pmf $p_{Y|X}(v|4)$.
Solution:

$$p_{Y|X}(v|4) = \frac{p_{X,Y}(4,v)}{p_X(4)} = \begin{cases} \frac{0.1}{0.25} = 0.4 & \text{if } v=1 \\ \frac{0.05}{0.25} = 0.2 & \text{if } v=2 \\ \frac{0.1}{0.25} = 0.4 & \text{if } v=3 \end{cases}$$

(d) Find $P\{X < Y\}$.
Solution:

$$P\{X < Y\} = p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(2,3) = 0.05.$$

2. **[10 points]** Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 1$ and $\sigma^2 = 4$. Find $P(|X| < 1)$. Express your solution in terms of the Q function.

Solution:

$$\begin{aligned} P(|X| < 1) &= P(-1 < X < 1) = P(X < 1) - P(X < -1) = \\ &= P\left(\frac{X-\mu}{\sigma} < \frac{1-1}{2}\right) - P\left(\frac{X-\mu}{\sigma} < \frac{-1-1}{2}\right) \\ &= P(Z < 0) - P(Z < -1), \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$. Using the symmetry properties of the Q function and the given hint:

$$P(|X| < 1) = Q(0) - Q(1) = \frac{1}{2} - Q(1).$$

3. **[6+8+12 points]** Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 1$ bus per minute. Bob arrives at the bus stop at 12 noon and wants to catch the next bus. For all parts of this question, you can leave your answer in terms of e, the base of natural logarithm, e.g. $2e^{-1}$.

   (a) What is the probability that exactly three buses arrive within one minute?
   Solution: Denote the number of buses arriving within one minute by N_1, which has a Poisson distribution with mean $\lambda \times 1 = 1$, hence

$$P(N_1 = 3) = \frac{e^{-1}1^3}{3!} = \frac{e^{-1}}{6}.$$

   (b) Given Bob is still waiting at the bus stop at 12:05pm (i.e., no bus arrived), what is the probability that he will still be waiting at 12:10pm?
   Solution: Denote the time between 12 noon and the arrival of the first bus by T_1, which has an exponential distribution with rate $\lambda = 1$, hence using memoryless property of exponential distribution,

$$P(T_1 > 10 | T_1 > 5) = P(T_1 > 5) = e^{-5}.$$

   (c) Given a total of two buses arrived between 12pm and 12:02pm, what's the probability that exactly one bus arrived between 12pm and 12:01pm?
   Solution: Denote the number of buses arriving between 12pm and 12:02pm by N_2, the number arriving between 12pm and 12:01pm by N_1 and the number arriving between 12:01pm and 12:02pm by M_1. The random variable N_2 has a Poisson distribution with mean $2\lambda = 2$. The random variables N_1 and M_1 are independent and each has a Poisson distribution with mean $\lambda = 1$. Hence,

$$P(N_1 = 1 | N_2 = 2) = \frac{P(N_1 = 1, M_1 = 1)}{P(N_2 = 2)} = \frac{P(N_1 = 1)P(M_1 = 1)}{P(N_2 = 2)} = \frac{e^{-1}e^{-1}}{\frac{e^{-2}2^2}{2!}} = 1/2.$$

4. **[8+12 points]** Suppose a random variable X has the following pdf:

$$f_X(u) = \begin{cases} c(1-u) & 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

   (a) Find the constant c
   Solution: The total probability should be equal to 1,

$$\int_{-\infty}^{\infty} f_X(u)du = c\int_0^1 (1-u)du = \frac{1}{2}c = 1$$

   Therefore, $c = 2$.

   (b) Let Y denote the standardized version of X, i.e., Y is a linearly scaled version of X that has mean 0 and variance 1. What is the PDF of Y?
   Solution: The mean of the random variable X

$$\mu = E[X] = 2\int_0^1 u(1-u)du = \frac{1}{3}$$

and

$$E[X^2] = 2\int_0^1 u^2(1-u)du = \frac{1}{6}$$

Therefore the standard deviation for X is $\sigma = \sqrt{E[X^2] - E[X]^2} = \frac{1}{3\sqrt{2}}$. The support for Y is therefore $[(0-1/3)3\sqrt{2}, (1-1/3)3\sqrt{2}] = [-\sqrt{2}, 2\sqrt{2}]$. Use the linear scaling property of the pdf that if $Y = aX + b$, then $f_Y(v) = \frac{1}{a} f_X((v-b)/a)$. Since $Y = \frac{X-\mu}{\sigma}$, $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$,

$$f_Y(v) = \sigma f_X(\sigma v + \mu) = \begin{cases} 2(1-(v/3\sqrt{2}+1/3))/3\sqrt{2} = \frac{2\sqrt{2}}{9} - \frac{1}{9}v & -\sqrt{2} \le v \le 2\sqrt{2} \\ 0 & \text{else} \end{cases}$$

5. **[14 points]** Let $X_1 \sim \text{Exp}(\lambda)$ and $X_2 \sim \text{Exp}(3\lambda)$ be independent random variables. Let also $Y = \min\{X_1, X_2\}$. Suppose that $Y = 3$ is observed. Find the Maximum Likelihood estimate of λ.

   Solution:

$$\begin{aligned} 1 - F_Y(y) = P(Y > y) &= P(\min\{X_1, X_2\} > y) = P(X_1 > y)P(X_2 > y) \\ &= e^{-\lambda y} e^{-3\lambda y} = e^{-4\lambda y}, \quad y \ge 0. \end{aligned}$$

   Therefore, $Y \sim \text{Exp}(4\lambda)$. Moreover,

$$f_Y(y) = 4\lambda e^{-4\lambda y}, \quad y \ge 0.$$

   For $Y = 3$, we have

$$L(\lambda) = f_Y(3) = 4\lambda e^{-12\lambda}.$$

   Setting the derivative with respect to λ of the likelihood function to zero and solving for λ we obtain:

$$\lambda_{\text{ML}} = \frac{1}{12}.$$

   Since the derivative reaches a maximum at $\lambda_{\text{ML}} = \frac{1}{12}$, this is the ML estimate.