University of Illinois Fall 2018

ECE 313: Hour Exam I

Wednesday, October 10, 2018
8:45 p.m. — 10:00 p.m.

1. **[6+8+10 points]** We have a bag of coins. Pick one coin from the bag and flip the same coin repeatedly.

   (a) The coin shows heads with probability p each time it is flipped. We flip the coin n times and denote the number of heads we observe by X. We will use $\hat{p} = \frac{X}{n}$ to estimate p. If we want to estimate p to within 0.1 with 99% confidence, how many times do we need to flip the coin?
   Solution: Let n be the number of flips and X be the total number of heads showing up. We can estimate p using $\hat{p} = \frac{X}{n}$. Using the result on confident interval for binomial distribution, we obtain that

   $$\begin{aligned} 1 - \frac{1}{a^2} &= 99\% \\ a &= 10 \end{aligned}$$

   The half-width of the confidence interval is $\frac{a}{2\sqrt{n}} = \frac{5}{\sqrt{n}}$, which should be less than or equal to 0.1. This requires $n \geq (\frac{5}{0.1})^2 = 2500$.

   (b) The coin shows heads with probability p each time it is flipped. We observe the second head at the sixth trial. Compute the ML estimate of p. Show your work.
   Solution: Let S_2 be the number of trials until the second head shows. S_2 has a negative binomial distribution with parameter p and $r = 2$. Hence, $P(S_2 = 6) = \binom{5}{1} p^2 (1-p)^4$. Differentiate $P(S_2 = 6)$ with respective to p and set the derivative to 0, we obtain

   $$\begin{aligned} 2p(1-p)^4 - 4p^2(1-p)^3 &= 0 \\ 2(1-p) - 4p &= 0 \\ p = \frac{1}{3} \end{aligned}$$

   Note that the ML estimate of p is the same if we observe a *total* of two heads out of six trials, regardless of at which trial a head shows up.

   (c) We flip the same coin three times and observe the total number of heads, X. Let H_0 be the hypothesis that the coin is fair. Let H_1 be the hypothesis that the coin is biased and shows heads with probability $\frac{1}{4}$. Write down the ML decision rule. Compute p_{miss}.
   Solution:

X	0	1	2	3
H_1	$\underline{(\frac{3}{4})^3}$	$\underline{3(\frac{1}{4})(\frac{3}{4})^2}$	$3(\frac{1}{4})^2(\frac{3}{4})$	$(\frac{1}{4})^3$
H_0	$(\frac{1}{2})^3$	$3(\frac{1}{2})^3$	$\underline{3(\frac{1}{2})^3}$	$\underline{(\frac{1}{2})^3}$

   $$p_{miss} = P(H_0|H_1) = 3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^3 = \frac{5}{32}.$$

2. **[10+10 points]** Suppose five cards are drawn from a standard 52 card deck of playing cards, with all possibilities being equally likely. (A standard card deck has four suites and each suite has numbers 1 to 13.) A "4 of a kind" is the event that four of the five cards have the same number. For the following questions, you may leave your answer in terms of binomial coefficients (without simplification.)

(a) What is the conditional probability of "4 of a kind", given that one of the five cards drawn is the Ace of Clubs and another is the Ace of Diamonds?

Solution: Given that two of the five cards drawn are 1C and 1D, the only way to form the "4 of a kind" is through Aces, in which case there are 12 choices for the number of the remaining card and 4 choices for the suit of that card. Therefore:

$$P(\text{“4 of a kind”} \,|\, \text{“two of the cards are 1C, 1D”}) = \frac{12 \times 4}{\binom{50}{3}} = \frac{48}{\binom{50}{3}}.$$

(b) What is the conditional probability of "4 of a kind", given that one of the five cards drawn is the Ace of Clubs and another is the Two of Clubs?

Solution: Given that two of the five cards drawn are 1C and 2C, there are two ways in which we can form the "4 of a kind": (1) Aces form the "4 of a kind", with the fifth card being 2C, and (2) Twos form the "4 of a kind", with the fifth card being 1C. Therefore:

$$P(\text{“4 of a kind”} \,|\, \text{“two of the cards are 1C, 2C”}) = \frac{1+1}{\binom{50}{3}} = \frac{2}{\binom{50}{3}}.$$

3. **[3+7 points]** Consider an s-t network with two links connected in parallel between the source and terminal. Each link has capacity 10. Link 1 fails with probability 0.4 and link 2 fails with probability 0.6. The links fail independently.

(a) What is the outage probability?

Solution:

$$P(\text{outage}) = P(\text{both links fail}) = 0.4 \times 0.6 = 0.24.$$

(b) We measure the network capacity and find it to be 10. What is the probability that link 1 failed?

Solution: Since the network capacity is 10, we know that exactly one link has failed. Let F_1 be the event that link 1 failed and F_2 be the event that link 2 failed. Let A be the event that the network capacity is 10.

$$\begin{aligned} P(F_1|A) &= \frac{P(F_1A)}{P(A)} = \frac{P(F_1A)}{P(F_1A) + P(F_1^cA)} \\ &= \frac{P(F_1F_2^c)}{P(F_1F_2^c) + P(F_1^cF_2)} \\ &= \frac{P(F_1)P(F_2^c)}{P(F_1)P(F_2^c) + P(F_1^c)P(F_2)} \\ &= \frac{0.4 \times 0.4}{0.4 \times 0.4 + 0.6 \times 0.6} = \frac{0.16}{0.52} = \frac{4}{13}. \end{aligned}$$

4. **[6+6 points]** Suppose A, B, and C are events for a probability experiment such that A and B are mutually independent, $P(A) = P(B) = P(C) = 0.4$, $P(AC) = 0.3$, $P(BC) = 0.2$, and $P(ABC) = 0.1$.

(a) What is $P(AB^cC)$?

Solution:

$$P(AB^cC) = P(AC) - P(ABC) = 0.3 - 0.1 = 0.2 \tag{1}$$

(b) What is $P(C|AB)$?

Solution:

$$P(C|AB) = \frac{P(ABC)}{P(AB)} = \frac{P(ABC)}{P(A)P(B)} = \frac{0.1}{0.4 \times 0.4} = \frac{0.1}{0.16} = \frac{5}{8}.$$

5. [**6+8+10+10 points**] The four parts are unrelated.

(a) Consider a random experiment where we roll two fair dice in each trial. Let the number showing on the first die be i, and the number showing on the second die be j. Define the random variable $X = \max\{i, j\}$. What is the expected number of trials until we observe $X = 2$?

Solution: From the description of the random experiment

$$P(X = 2) = P(\{(1,2),(2,1),(2,2)\}) = \frac{3}{36}.$$

The number of trials until we observe $X = 2$ is a geometric random variable Y with parameter $p = P(X = 2)$. Therefore,

$$E[Y] = \frac{1}{p} = \frac{36}{3} = 12.$$

(b) Consider the experiment of tossing a fair coin 6 times. Let X denote the number of heads among the 6 tosses and Y be the number of heads in the first 3 tosses. Define the random variable $Z = X - Y$. Compute the probability $P(Z = 2)$.

Solution: Let $Z_1, Z_2, \ldots, Z_6$ be independent Bernoulli random variables taking the value 1 when a toss is heads and 0 otherwise. Then,

$$X = Z_1 + X_2 + \cdots + Z_6 \sim \text{Bin}(6, 1/2)$$

and

$$Y = Z_1 + Z_2 + Z_3 \sim \text{Bin}(3, 1/2).$$

Therefore, $Z = Z_4 + Z_5 + Z_6 \sim \text{Bin}(3, 1/2)$. Hence,

$$P(Z = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{8}.$$

(c) Suppose four people write their first names (Alex, Bob, Chris, Alex) on slips of paper. Note that two people have the same first names. The slips of paper are randomly shuffled and then each person gets back one slip of paper; all possibilities of who gets which slip are equally likely. Let X denote the number of people who get back the slip with their first name (i.e. the number of matches). Find the pmf of X.

Solution: We use the initial of the first names A, B, C, A. Then the whole sample space for the experiment is all possible ways of arranging those four letters, e.g. ABCA, ACBA, BACA, ... Since two letters are indistinguishable, the number of elements should be $\frac{4!}{2!} = 12$.

Possible values of X are 0, 1, 2, 4. When the observed sequences are $BAAC$ and $CAAB$, $X = 0$. So $p_X(0) = \frac{2}{12} = \frac{1}{6}$. When the observed sequences are $AABC$, $BCAA$, $ACAB$, and $CABA$, $X = 1$. If the sequences are $ACBA$, $ABAC$, $BACA$, $CBAA$, and $AACB$, then we get $X = 2$. When the sequence is $ABCA$, we get complete match. Therefore, $X = 4$.

$$p_X(0) = \frac{1}{6}, \quad p_X(1) = \frac{1}{3}, \quad p_X(2) = \frac{5}{12}, \quad p_X(4) = \frac{1}{12} \tag{2}$$

(d) There are m kids, and there are n different brands of chocolate. Each kid chooses one brand of chocolate independently at random. (More than one kid can choose the same type of chocolate.) What is the expected number of different chocolate brands that are selected?
(**Hint**: It may be easier to first compute the expected number of chocolate brands that are **not** selected.)

Solution: Let X_k be a random variable taking the value 1 if the kth brand (or equivalently the kth box) is not selected by any of the kids and 0 otherwise. Since the choices of different kids are independent, there are $(n-1)^m$ m-tuples of brand choices not containing the kth brand and n^m possible m-tuples of brand choices in total. Hence,

$$P(X_k = 1) = \left(\frac{n-1}{n}\right)^m .$$

Therefore, the expected number of chocolate brands that will not be selected is

$$E[X] = \sum_{k=1}^{n} E[X_k] = \sum_{k=1}^{n} P(X_k = 1) = \sum_{k=1}^{n} \left(\frac{n-1}{n}\right)^m = n\left(\frac{n-1}{n}\right)^m ,$$

which leads to $E[\tilde{X}] = n\left[1 - \left(\frac{n-1}{n}\right)^m\right]$. Here, $\tilde{X}$ is the number of chocolate brands that will be selected by the kids.

Alternatively, the expected number of chosen chocolate brands can be computed directly by noting that

$$E[\tilde{X}] = \sum_{k=1}^{n} P(X_k = 0) = n\left[1 - \left(\frac{n-1}{n}\right)^m\right] .$$