University of Illinois Fall 2017

ECE 313: Hour Exam II

Wednesday, November 15, 2017
8:45 p.m. — 10:00 p.m.

1. **[6 points]** Suppose X and Y have the joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} C & u^2+v^2 \leq 1 \\ 0 & \text{else,} \end{cases}$$

   (a) Find C.
   Solution: Since the pdf is uniform and the support has area π, $C = \frac{1}{\pi}$.

   (b) Are X and Y independent? Explain why.
   Solution: No, they are not since the support is not a product set.

2. **[10 points]** Suppose that buses are scheduled to arrive at a bus stop at noon but are always X minutes late, where X is an exponential random variable with rate $\lambda = 1/5$. Suppose that you arrive at the bus stop precisely at noon. *NOTE:* All answers can be left expressed in terms of exponentials, e.g., ae^{-b}.

   (a) Compute the probability that you have to wait for more than 10 minutes for the bus to arrive.
   Solution: We use the complimentary CDF for exponential distribution,

$$P(X \geq 10) = e^{-10\lambda} = e^{-2}. \quad (1)$$

   (b) Suppose that you have already waited for 10 minutes. Compute the probability that you have to wait an additional 2 minutes or more.
   Solution: We want $P(X \geq 12|X \geq 10)$. Use the memoryless property of the exponential distribution,

$$P(X \geq 12|X \geq 10) = P(X \geq 2) = e^{-2\lambda} = e^{-\frac{2}{5}}. \quad (2)$$

3. **[16 points]** The number of failures occurring in a particular wireless network over the time interval $[0,t)$ can be modeled as a Poisson process $\{N(t), t \geq 0\}$. On average, there is a failure every 4 days, i.e., the intensity of the process is $\lambda = 0.25$/day. *NOTE:* All answers can be left expressed in terms of sums of exponentials, e.g., $ae^{-b} + ce^{-d}$.

   (a) What is the probability of at most 1 failure in $[0,8)$ and at least 2 failures in $[8,16)$? The given time intervals are in days.
   Solution: The probability is

$$p = P\left(N(8) - N(0) \leq 1, N(16) - N(8) \geq 2\right).$$

   By the independence of increments of a Poisson process, we have:

$$\begin{aligned} p &= P\left(N(8) - N(0) \leq 1\right) P\left(N(16) - N(8) \geq 2\right) \\ &= P(N(8) \leq 1) P(N(8) \geq 2). \end{aligned}$$

$$P(N(8) \le 1) = P(N(8) = 0) + P(N(8) = 1) = e^{-0.25 \cdot 8} + 0.25 \cdot 8 \cdot e^{-0.25 \cdot 8} = 3e^{-2}.$$
$$P(N(8) \ge 2) = 1 - P(N(8) \le 1) = 1 - 3e^{-2}.$$

Putting all the above results together:

$$p = 3e^{-2}(1 - 3e^{-2}) = 3e^{-2} - 9e^{-4}.$$

(b) Let T_3 be the time of the third failure. Compute $P(T_3 > 8)$ (time unit: days).
Solution:

$$P(T_3 > 8) = P(N(8) \le 2) = e^{-0.25 \cdot 8} \left(\sum_{n=0}^{2} \frac{(0.25 \cdot 8)^n}{n!} \right) = 5e^{-2}.$$

4. **[16 points]** Suppose the output transmission power of a cellular phone is X dBm (decibel-milliwatts), where X is uniformly distributed over the interval $[20, 30]$. Then $Y = 10^{X/10}$ is the transmission power in mW (milliwatts). Find the pdf of Y.
NOTE: The answer can be left expressed in terms of logarithms like $\log_{10} e$ or $\ln 10$.

Solution: The support of Y is $[10^{20/10}, 10^{30/10}]$ or $[100, 1000]$. Then

$$\begin{aligned} F_Y(c) &= P\{10^{X/10} \le c\} \\ &= P\{X \le 10 \log_{10} c\} \\ &= \begin{cases} 0 & \text{if } c < 100 \\ \frac{10 \log_{10} c - 20}{30 - 20} & \text{if } 100 \le c \le 1000 \\ 1 & \text{if } c > 1000 \end{cases} \\ &= \begin{cases} 0 & \text{if } c < 100 \\ \log_{10} c - 2 & \text{if } 100 \le c \le 1000 \\ 1 & \text{if } c > 1000 \end{cases}. \end{aligned}$$

The pdf of Y is

$$f_Y(c) = F'_Y(c) = \begin{cases} \frac{\log_{10} e}{c} & \text{if } 100 \le c \le 1000 \\ 0 & \text{else} \end{cases}.$$

5. **[22 points]** The pdf of the Kumaraswamy distribution is

$$p_W(w) = abw^{a-1}(1 - w^a)^{b-1}, \text{ where } w \in [0, 1],$$

where a and b are non-negative shape parameters. We observe a signal in the presence of additive Kumaraswamy noise with $a = 2$ and $b = 2$, i.e., $Y = X + W$, where Y is the observation, X is the original signal and W is the noise. Note that W is supported on the interval $[0, 1]$ and is not symmetric around $1/2$.

(a) Suppose we have two possible signals $X \in \{0, 2\}$, where $P(X = 0) = 0.3$ and $P(X = 2) = 0.7$. Design a decision rule that minimizes the probability of error.
Solution: We notice that the conditional probability distributions for the two signals do not overlap, as one has support on the interval $[0, 1]$ and the other on the interval $[2, 3]$. Thus any threshold-based test with a threshold in the non-overlapping region would minimize error probability. As a particular example, letting the channel output be y and the decision $\hat{x}$, an optimal rule is:

$$y \underset{\hat{x}=2}{\overset{\hat{x}=0}{\lesseqgtr}} 1.5.$$

(b) Evaluate the probability of error, which we denote by $P_e^{(b)}$.

Solution: $P_e^{(b)} = 0$, clearly there is no error, since there is no confusion.

(c) Suppose we instead have two possible signals $X \in \{0, \frac{1}{2}\}$, where $P(X = 0) = 0.5$ and $P(X = \frac{1}{2}) = 0.5$. Design a decision rule that minimizes the probability of error.

Solution: Under one hypothesis, we have a conditional output distribution of $4y(1-y^2)$ and in the other, we have a conditional output distribution of $4(y-\frac{1}{2})(1-(y-\frac{1}{2})^2)$. Since the two messages are equiprobable, we just need to compare the likelihoods. Moreover, we can sketch and observe we just need to find the crossing point to get a threshold test. Let us equate to find the threshold.

$$\begin{aligned}
4y(1-y^2) &= 4(y-\tfrac{1}{2})(1-(y-\tfrac{1}{2})^2) \\
y(1-y^2) &= (y-\tfrac{1}{2})(1-(y^2-y+\tfrac{1}{4})) \\
y-y^3 &= (y-\tfrac{1}{2})(-y^2+y+\tfrac{3}{4}) \\
y-y^3 &= -y^3+y^2+\tfrac{3}{4}y+\tfrac{1}{2}y^2-\tfrac{1}{2}y-\tfrac{3}{8} \\
y &= \tfrac{3}{2}y^2+\tfrac{1}{4}y-\tfrac{3}{8} \\
0 &= \tfrac{3}{2}y^2-\tfrac{3}{4}y-\tfrac{3}{8} \\
0 &= 3y^2-\tfrac{3}{2}y-\tfrac{3}{4} \\
0 &= y^2-\tfrac{1}{2}y-\tfrac{1}{4}.
\end{aligned}$$

Now using the quadratic formula, we get the following roots of the equation.

$$\frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4 \cdot -\frac{1}{4}}}{2} = \frac{1}{2}\left(\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right)$$

Since the noise and signaling are positive-valued, we care about the positive root for our threshold. This is

$$\frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{4}.$$

Thus our decision rule is

$$y \underset{\hat{x}=1/2}{\overset{\hat{x}=0}{\lessgtr}} \frac{1+\sqrt{5}}{4}.$$

(d) Let us call the average probability of error in this case to be $P_e^{(d)}$. Is $P_e^{(d)} \geq P_e^{(b)}$, YES or NO? (Note: You do NOT need to compute $P_e^{(d)}$.)

Solution: YES, clearly there will be some positive error probability, since there is overlap. This is more than zero.

6. **[16 points]** We have a coin with an unknown probability of showing head. We denote this unknown probability by X and we know that the pdf of X is given by

$$f_X(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)},$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, and $\Gamma(n) = (n-1)!$ if n is a positive integer. We toss the coin 5 times. Let $\alpha = 2$ and $\beta = 2$. What is the probability that we observe 4 heads?

Solution: Let Y be the number of heads we obtain. $p_{Y|X}(v|p) \sim$ Binomial(5,p).

$$\begin{aligned}
P(Y=4) &= \int_0^1 f_X(p)p_{Y|X}(v|p)dp \\
&= \int_0^1 \frac{p(1-p)}{B(2,2)}\binom{5}{4}p^4(1-p)dp \\
&= 30\int_0^1 p^5(1-p)^2 dp \\
&= 30B(6,3) = 30\frac{\Gamma(6)\Gamma(3)}{\Gamma(9)} = 30\frac{5!2!}{8!} = \frac{5}{28}.
\end{aligned}$$

Note $B(2,2) = \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{1}{3!}$ and $\int_0^1 \frac{p^4(1-p)^2}{B(5,3)}dp = 1$.

7. **[14 points]** You have two machines. Machine 1 has lifetime T_1, which is Exponential(λ_1), and Machine 2 has lifetime T_2, which is Exponential(λ_2). The lifetimes are independent random variables. Machine 1 starts at time 0 and Machine 2 starts at time T. Assume that T is deterministic. Compute the probability that Machine 1 is the first to fail.

Solution: We note that

$$\begin{aligned}
P(T_1 < T_2 + T) &= P(T_1 < T) + P(T_1 \geq T, T_1 < T_2 + T) \\
&= P(T_1 < T) + P(T_1 < T_2 + T | T_1 \geq T)P(T_1 \geq T) \\
&= 1 - e^{-\lambda_1 T} + P(T_1 < T_2)e^{-\lambda_1 T} \\
&= 1 - e^{-\lambda_1 T} + \frac{\lambda_1}{\lambda_1 + \lambda_2}e^{-\lambda_1 T}.
\end{aligned}$$

Here, the memoryless property of the exponential distribution has been used. Also, for $P(T_1 < T_2)$ the following computation has been employed:

$$\begin{aligned}
P(T_1 < T_2) &= \int_0^{+\infty}\int_0^v \lambda_1 e^{-\lambda_1 u}\lambda_2 e^{-\lambda_2 v} du dv = \int_0^\infty \lambda_2 e^{-\lambda_2 v} F_{T_1}(v) dv \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 v}\left(1 - e^{-\lambda_1 v}\right) dv = 1 - \int_0^\infty \lambda_2 e^{-(\lambda_1 + \lambda_2)v} dv \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$