University of Illinois Fall 2016

ECE 313: Conflict Final Exam

Tuesday, December 13, 2016
7 p.m. — 10 p.m.
ECEB 1013

Name: (in BLOCK CAPITALS) ________________________________

NetID: ________________________________

Signature: ________________________________

Section:
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 11 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 14 points ____________
2. 14 points ____________
3. 20 points ____________
4. 14 points ____________
5. 20 points ____________
6. 22 points ____________
7. 18 points ____________
8. 18 points ____________
9. 12 points ____________
10. 18 points ____________
11. 30 points ____________

Total (200 points) ____________

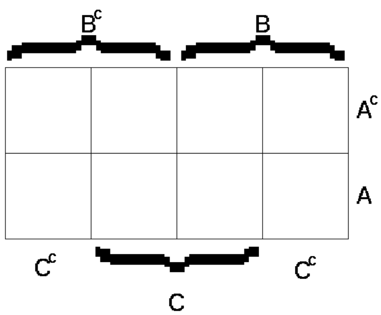
1. [**14 points**] A drawer contains 4 black, 6 red, and 8 yellow socks. Two socks are selected at random from the drawer.

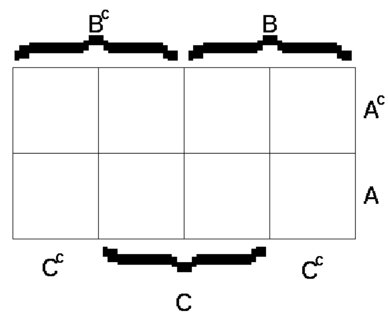
   (a) What is the probability the two socks are of the same color?

   (b) What is the conditional probability both socks are yellow given they are of the same color?

2. [**14 points**] The two parts of this problem are unrelated.

(a) Suppose A, B, and C are events for a probability experiment such that B and C are mutually independent, $P(A) = P(B^c) = P(C) = 0.5$, $P(AB) = P(AC) = 0.3$, and $P(ABC) = 0.1$. Fill in the probabilities of all events in a Karnaugh map. Show your work but use the map on the right to depict your final answer.





(b) Let A, B be two disjoint events on a sample space Ω. Obtain a formula for the probability of A occurring before B in an infinite sequence of independent trials.

3. **[20 points]** Bob performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of three fair coins.

   (a) Given that Bob has just had a trial with all 3 coins landing on tails, what is the probability that both of the next two trials will also have this result?

   (b) Whenever all three coins land on the same side in any given trial, Bob calls the trial a success. Obtain the pmf for K, the number of trials up to, but not including, the second success.

   (c) Alice conducts an experiment like Bob's, except that she uses 4 coins for the first trial, and then she obeys the following rule: Whenever all of the coins land on the same side in a trial, Alice permanently removes one coin from the experiment and continues with the trials. She follows this rule until the third time she removes a coin, at which point the experiment ceases. Obtain $E[N]$, where N is the number of trials in Alices experiment.

4. **[14 points]** Suppose S and T represent the lifetimes of two phones, the lifetimes are independent, and each has the exponential distribution with parameter $\lambda = 1$.

   (a) Obtain $P\{|S - T| \leq 1\}$.

   (b) Let $Z = (S - 1)^2$. Obtain $f_Z(c)$, the pdf of Z, for all c.

5. [**20 points**] An X-ray source transmits photons towards a detector at mean rate $\lambda = 8$ photons/msec according to a Poisson process. Suppose each photon transmitted is detected independently with probability $p = 0.25$.

   (a) What is the pdf of the time the fifth photon is transmitted (measured in msec)?

   (b) Given n photons are transmitted in a certain period, what is the probability k photons are detected. (Assume $n \geq 1$ and $0 \leq k \leq n$.)

   (c) What is the pmf for the number of photons *detected* in an interval of duration 10 msec? Simplify your answer as much as possible. (Hint: Let X denote the number of photons transmitted and Y the number detected. Obtain the pmf of Y using the law of total probability.)

6. [**22 points**] Suppose that a point (X, Y) is picked uniformly in the triangle $\{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$.

(a) Obtain $f_{X,Y}(u, v)$, the joint pdf of X and Y, for all u and v.

(b) Obtain $f_{Y|X}(v|u)$, the conditional pdf of Y given X, for all u and v.

(c) Calculate $E[Y|X = u]$.

(d) Compute $E[(X - Y)^2|X = u]$.

7. [**18 points**] Suppose two different data streams, S_1 and S_2, share a communication channel. S_1 transmits on any given day with probability $\frac{2}{3}$, while S_2 transmits on the other days. Let X be the number of bits per hour sent over the channel . If S_1 transmits, X is a geometric random variable with parameter $p = \frac{2}{3}$, whereas if S_2 transmits, X is a geometric random variable with parameter $p = \frac{1}{2}$. You have no way to find out who is is trasnmitting except by observing the number of bits sent.

   (a) Suppose you use the following rule to decide who is trasnmitting today, based on observation of the data rate for one hour, X:

   - If $X > 1$, say S_1
   - If $X = 1$, say S_2

   What is the probability that this rule makes an error?

   (b) The maximum likelihood decision rule for this hypothesis testing problem can be stated as:

   - If $X \geq k_{ML}$, say S_2 transmits.
   - Otherwise, say S_1 transmits.

   for some positive integer value k_{ML}. Obtain the value of k_{ML}.

8. **[18 points]** Suppose that X, Y are jointly Gaussian random variables with $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. Let their correlation coefficient be ρ with $|\rho| < 1$. Based on (X, Y), we define the following random variables:

$$W = \left(\frac{1}{2\sqrt{1+\rho}} + \frac{1}{2\sqrt{1-\rho}}\right) X + \left(\frac{1}{2\sqrt{1+\rho}} - \frac{1}{2\sqrt{1-\rho}}\right) Y$$
$$Z = \left(\frac{1}{2\sqrt{1+\rho}} - \frac{1}{2\sqrt{1-\rho}}\right) X + \left(\frac{1}{2\sqrt{1+\rho}} + \frac{1}{2\sqrt{1-\rho}}\right) Y$$

(a) Are W, Z jointly Gaussian? Justify your answer.

(b) Obtain $f_{W,Z}(w, z)$.

(c) Obtain the MMSE estimator of Z given W.

(d) Obtain the linear MMSE estimator of X given W.

9. [**12 points**] Let $U_1, \ldots, U_n$ be independent, exponentially distributed random variables with unknown parameter λ.

   (a) Identify the ML estimator $\widehat{\lambda}$ for joint observations $U_1, \ldots, U_n$.

   (b) Using the Chebychev inequality, identify a number of observations n large enough so that $[(0.9)\widehat{\lambda}_{ML}, (1.1)\widehat{\lambda}_{ML}]$ is a confidence interval for estimation of λ with confidence level 96%.

10. **[18 points]** Suppose U and V are independent random variables such that U is uniformly distributed over $[0, 1]$ and V is uniformly distributed over $[0, 2]$. Let $S = U + V$.

   (a) Obtain the mean and variance of S.

   (b) Derive and carefully sketch the pdf of S.

   (c) Obtain $\widehat{E}[U|S]$, the minimum mean square error linear estimator of U given S.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose X and Y are jointly continuous-type random variables with finite variance.

TRUE	FALSE	
☐	☐	If the MMSE for estimating Y from X is $\mathrm{Var}(Y)$, then X and Y must be uncorrelated.
☐	☐	If X and Y are uncorrelated then the MMSE for estimating Y from X is $\mathrm{Var}(Y)$.

(b) Suppose Y is a nonnegative random variable with $E[Y] = 10$, and X is a random variable with mean 10 and variance 16.

TRUE	FALSE	
☐	☐	It is possible that the standard deviation of Y is 10.
☐	☐	It is possible that $P\{Y \geq 30\} = 1/4$.
☐	☐	It is possible that $P\{X \geq 0\} = 0.5$.

(c) Suppose X and Y are two Binomial random variables with parameters n_X, p_X, and n_Y, p_Y, respectively.

TRUE	FALSE	
☐	☐	If $Y = n_X - X$, then $p_y(k) = p_x(n_X - k)$.
☐	☐	If $n_X = n_Y > 20$ and $p_x(1) > p_y(1)$ then $E[X] > E[Y]$.
☐	☐	If $Z = X + Y$, then Z is $Binomial(n_X + n_Y, p_X + p_Y)$.

(d) Let A, B be nonempty events in a sample space Ω. Assume that $E_1, E_2, \ldots, E_n$ is a partition of Ω.

TRUE	FALSE	
☐	☐	If A, B are mutually exclusive, then $P(A\|B) = P(A)$.
☐	☐	Suppose that $A \neq B$. Then $\sum_{i=1}^{n} P(E_i\|A) \neq \sum_{i=1}^{n} P(E_i\|B)$.