University of Illinois Fall 2016

ECE 313: Exam II

Wednesday, November 9, 2016
8:45 p.m. — 10:00 p.m.
A-C will go to room ECEB 1013
D-I will go to room ECEB 1015
J-Z will go to room ECEB 1002

1. **[20 points]** Let X be a random variable with CDF given by

$$F_X(u) = \begin{cases} C_1 & u < -3 \\ \frac{2}{6}u + 1 & -3 \leq u \leq -1 \\ C_2 & -1 < u < 0 \\ \frac{1}{6}u + \frac{4}{6} & 0 \leq u < 2 \\ C_3 & else. \end{cases}$$

   (a) Obtain the values of the constants C_1, C_2, and C_3.
   Solution: By properties of the CDF, $\lim_{u\to-\infty} F(u) = 0$, which implies that $C_1 = 0$, Similarly, $\lim_{u\to\infty} F(u) = 1$, which implies that $C_3 = 1$. Also, $F_X(u)$ must be nondecreasing, which along with the values of $F_X(-1) = F_X(0) = \frac{4}{6}$, implies that $C_2 = \frac{4}{6} = \frac{2}{3}$.

   (b) Obtain $P\{X = 1.5\}$.
   Solution: There is no discontinuity (jump) at $X = 1.5$, hence $P\{X = 1.5\} = 0$.

   (c) Obtain $P\{X > 1\}$.
   Solution: $P\{X > 1\} = 1 - F_X(1) = 1 - \left(\frac{1}{6}(1) + \frac{4}{6}\right) = \frac{1}{6}$.

   (d) Obtain $f_X(u)$, the pdf of X, for all u.
   Solution: The pdf of X is the derivative of its CDF, hence,

$$f_X(u) = \frac{d}{du}F_X(u) = \begin{cases} \frac{2}{6} = \frac{1}{3} & -3 \leq u \leq -1 \\ \frac{1}{6} & 0 \leq u < 2 \\ 0 & else. \end{cases}$$

   (e) Obtain the median of X. Recall that the median is the value a such that $P\{X \leq a\} \geq 0.5$ and $P\{X \geq a\} \geq 0.5$
   Solution: The median can be obtained by setting $F_X(u) = 0.5$ because the CDF is continuous near its value of 0.5, which occurs between -3 and -1. Solving $\frac{2}{6}u + 1 = \frac{1}{2}$ for u yields $u = -\frac{3}{2}$.

2. **[14 points]** The random variable X has the $N(-2, 16)$ distribution. Express the answers to the following questions in terms of the Φ function.

   (a) Obtain $P\{X^2 - 2X \geq 3\}$.
   Solution: $X^2 - 2X \geq 3$ when $X \leq -1$ or $X \geq 3$

$$\begin{aligned} P\{X^2 - 2X \geq 3\} &= P\{X \leq -1\} + P\{X \geq 3\} \\ &= P\left\{\frac{X+2}{4} \leq \frac{1}{4}\right\} + P\left\{\frac{X+2}{4} \geq \frac{5}{4}\right\} = \Phi\left(\frac{1}{4}\right) + 1 - \Phi\left(\frac{5}{4}\right). \end{aligned}$$

(b) Let $Y = \frac{1}{2}X - 2$, obtain $P\{0 < Y < 2\}$.
Solution: The mean of Y is $E[Y] = E\left[\frac{1}{2}X - 2\right] = \frac{1}{2}E[X] - 2 = \frac{1}{2}(-2)(-2) = -3$, and the variance is $Var(Y) = Var\left(\frac{1}{2}X - 2\right) = \left(\frac{1}{2}\right)^2 Var(X) = \frac{1}{4}(16) = 4$, so $\frac{Y+3}{2}$ has standard normal distribution.

$$P\{0 < Y < 2\} = P\left\{\frac{3}{2} < \frac{Y+3}{2} < \frac{5}{2}\right\} = \Phi\left(\frac{5}{2}\right) - \Phi\left(\frac{3}{2}\right).$$

3. **[14 points]** Consider a binary hypothesis testing problem where

$$H_0: \quad f_0(y) = \begin{cases} C_1 e^{-y^2/2}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$
$$H_1: \quad f_1(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases},$$

where C_1 is some non-negative constant. It is known that $\pi_1 = C_1 \pi_0$.

(a) Determine the MAP decision rule. Draw a picture showing the rule.
Solution: We only need to consider the case $y \geq 0$ because that is the support of both pdfs. For the MAP decision rule, we decide H_1 when

$$\Lambda(y) = \frac{f_1(y)}{f_0(y)} \geq \frac{\pi_0}{\pi_1},$$

which in this case yields

$$\frac{e^{-y}}{C_1 e^{-y^2/2}} = \frac{1}{C_1} e^{-y+y^2/2} \geq \frac{1}{C_1}.$$

By canceling C_1 and taking natural logarithm on both sides, and multiplying by 2, the test can be simplified to

$$y^2 - 2y \geq 0$$

Hence, the MAP rule decides H_1 when $y \geq 2$ and H_0 when $0 \leq y < 2$.

(b) Obtain the probability of false alarm, $p_{\text{false alarm}}$, for the MAP rule. You can express it in terms of the constant C_1 and the Q function, the complementary CDF of a standard Gaussian.(**Hint**: Express $f_0(y)$ in terms of the density of $\mathcal{N}(0,1)$).
Solution: Note that for $y \geq 0$

$$f_0(y) = \sqrt{\frac{2}{\pi}} e^{-y^2/2} = 2\frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Hence, for p_{FA}, we have:

$$p_{FA} = P\left(\text{decide } H_1 | H_0\right) = P\left(y \geq 2 | H_0\right) = \int_2^\infty f_0(y)dy = 2\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 2Q(2)$$

4. **[16 points]** The lifetime T of a critical laser in a fiber optic modem is assumed to have a failure rate function $h(t)$, where time is measured in days of operation time.

(a) Find the mean lifetime, $E[T]$, in case $h(t) = \begin{cases} 0 & 0 \leq t \leq 1000 \\ 0.002 & t \geq 1000 \end{cases}$.

Solution: Since $h(t) = 0$ for the first 1000 days, the laser is still working after 1000 days with probability one. After that, the failure rate is constant, so $T - 1000$ has the exponential distribution with parameter 0.002, or mean 500. Thus, $E[T - 1000] = 500$, or $E[T] = 1500$ days.

(b) Suppose instead $h(t) = \alpha t$ for some $\alpha > 0$. What value of α gives $P\{T \geq 500\} = 90\%$?

Solution: We have $P\{T \geq 500\} = \exp\left(-\int_0^{500} h(s)ds\right) = \exp\left(-\frac{\alpha(500)^2}{2}\right)$. So α is determined by $-\frac{\alpha(500)^2}{2} = \ln(0.9)$ or $\alpha = \frac{-2\ln(0.9)}{(500)^2}$. (Since $-\ln(1-\epsilon) \approx \epsilon$ for small ϵ, $\alpha \approx \frac{0.2}{(500)^2} = 0.8 \times 10^{-6}$.)

5. **[18 points]** Suppose X and Y are random variables, each with values in $\{1, 2, 3\}$, and joint pmf given by:

	$X = 1$	$X = 2$	$X = 3$
$Y = 3$	$5c$	$6c$	$7c$
$Y = 2$	$4c$	$5c$	$6c$
$Y = 1$	$3c$	$4c$	$5c$

(a) Find the constant c.

Solution: Since the sum of all probabilities is $45c$, it must be that $c = \frac{1}{45}$.

(b) Find the pmf of X.

Solution: $P\{X = 1\} = \frac{3+4+5}{45} = \frac{4}{15}$, $P\{X = 2\} = \frac{4+5+6}{45} = \frac{1}{3}$, $P\{X = 3\} = \frac{5+6+7}{45} = \frac{2}{5}$.

(c) Find $P(X = Y | X \leq Y)$.

Solution: $P(X = Y | X \leq Y) = \frac{P\{(X=Y)\cap(X\leq Y)\}}{P\{X \leq Y\}} = \frac{P\{X=Y\}}{P\{X\leq Y\}} = \frac{3+5+7}{3+4+5+5+6+7} = \frac{15}{30} = \frac{1}{2}$.

6. **[18 points]** The two parts of this problem are unrelated.

(a) Let $S = X + Y$, where X and Y are independent random variables, X is a Bernoulli random variable with parameter p with $0 < p < 1$, and Y is a geometric random variable with parameter q with $0 < q < 1$. Find $P\{S = 3\}$.

Solution: From the convolution formula for independent random variables, or equivalently, the law of total probability,
$P\{S = 3\} = P\{X = 0\}P\{Y = 3\} + P\{X = 1\}P\{Y = 2\} = (1-p)q(1-q)^2 + pq(1-q)$.

(b) Let X and Y be two continuous-type random variables with joint pdf $f_{X,Y}(u, v) = \frac{1}{2}e^{-u}$ for $u \geq 0$ and $0 \leq v \leq 2$, and 0 otherwise. Let $Z = X + Y$. Find the pdf of Z.

Solution: X and Y are independent so the pdf of Z can be obtained through convolution of the pdfs of X and Y.
For $\alpha \in (0, 2)$, $f_Z(\alpha) = \int_0^\alpha \frac{1}{2}e^{u-\alpha}du = \frac{1}{2}(1 - e^{-\alpha})$.
For $\alpha \geq 2$, $f_Z(\alpha) = \int_0^2 \frac{1}{2}e^{u-\alpha}du = \frac{e^{-\alpha}}{2}(e^2 - 1)$.

Therefore, $$\boxed{f_Z(\alpha) = \begin{cases} 0 & \alpha \leq 0, \\ \frac{1}{2}(1 - e^{-\alpha}) & \alpha \in (0, 2), \\ \frac{e^{-\alpha}}{2}(e^2 - 1) & \alpha \geq 2. \end{cases}}$$