University of Illinois Fall 2016

ECE 313: Exam II

Wednesday, November 9, 2016
8:45 p.m. — 10:00 p.m.
A-C will go to room ECEB 1013
D-I will go to room ECEB 1015
J-Z will go to room ECEB 1002

Name: (in BLOCK CAPITALS) ______________________________

NetID: ______________________________

Signature: ______________________________

Section:
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that one 8.5”×11” sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of 6 problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 20 points ________
2. 14 points ________
3. 14 points ________
4. 16 points ________
5. 18 points ________
6. 18 points ________

Total (100 points) ________

1. **[20 points]** Let X be a random variable with CDF given by

$$F_X(u) = \begin{cases} C_1 & u < -3 \\ \frac{2}{6}u + 1 & -3 \le u \le -1 \\ C_2 & -1 < u < 0 \\ \frac{1}{6}u + \frac{4}{6} & 0 \le u < 2 \\ C_3 & \textit{else.} \end{cases}$$

(a) Obtain the values of the constants C_1, C_2, and C_3.

(b) Obtain $P\{X = 1.5\}$.

(c) Obtain $P\{X > 1\}$.

(d) Obtain $f_X(u)$, the pdf of X, for all u.

(e) Obtain the median of X. Recall that the median is the value a such that $P\{X \le a\} \ge 0.5$ and $P\{X \ge a\} \ge 0.5$

2. **[14 points]** The random variable X has the $N(-2, 16)$ distribution. Express the answers to the following questions in terms of the Φ function.

   (a) Obtain $P\{X^2 - 2X \geq 3\}$.

   (b) Let $Y = \frac{1}{2}X - 2$, obtain $P\{0 < Y < 2\}$.

3. **[14 points]** Consider a binary hypothesis testing problem where

$$H_0: \quad f_0(y) = \begin{cases} C_1 e^{-y^2/2}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$H_1: \quad f_1(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases},$$

where C_1 is some non-negative constant. It is known that $\pi_1 = C_1 \pi_0$.

(a) Determine the MAP decision rule. Draw a picture showing the rule.

(b) Obtain the probability of false alarm, $p_{\text{false alarm}}$, for the MAP rule. You can express it in terms of the constant C_1 and the Q function, the complementary CDF of a standard Gaussian.(**Hint**: Express $f_0(y)$ in terms of the density of $\mathcal{N}(0,1)$).

4. **[16 points]** The lifetime T of a critical laser in a fiber optic modem is assumed to have a failure rate function $h(t)$, where time is measured in days of operation time.

   (a) Find the mean lifetime, $E[T]$, in case $h(t) = \begin{cases} 0 & 0 \leq t \leq 1000 \\ 0.002 & t \geq 1000 \end{cases}$.

   (b) Suppose instead $h(t) = \alpha t$ for some $\alpha > 0$. What value of α gives $P\{T \geq 500\} = 90\%$?

5. **[18 points]** Suppose X and Y are random variables, each with values in $\{1, 2, 3\}$, and joint pmf given by:

$Y = 3$	$5c$	$6c$	$7c$
$Y = 2$	$4c$	$5c$	$6c$
$Y = 1$	$3c$	$4c$	$5c$
	$X = 1$	$X = 2$	$X = 3$

(a) Find the constant c.

(b) Find the pmf of X.

(c) Find $P(X = Y | X \leq Y)$.

6. **[18 points]** The two parts of this problem are unrelated.

(a) Let $S = X + Y$, where X and Y are independent random variables, X is a Bernoulli random variable with parameter p with $0 < p < 1$, and Y is a geometric random variable with parameter q with $0 < q < 1$. Find $P\{S = 3\}$.

(b) Let X and Y be two continuous-type random variables with joint pdf $f_{X,Y}(u, v) = \frac{1}{2}e^{-u}$ for $u \geq 0$ and $0 \leq v \leq 2$, and 0 otherwise. Let $Z = X + Y$. Find the pdf of Z.