University of Illinois Fall 2016

ECE 313: Exam I

Wednesday, October 5, 2016
8:45 p.m. — 10:00 p.m.
A-C will go to room ECEB 1013
D-I will go to room ECEB 1015
J-Z will go to room ECEB 1002

1. (a) The probability mass function should sum up to 1. Therefore,

$$\sum_{k=1}^{3} ck^2 = c(1+4+9) = 1, \quad \Rightarrow c = \frac{1}{14}$$

(b)

$$P\{X=3\} = c3^2 = c9 = \frac{9}{14}$$

(c)

$$E[X] = \sum_{k=1}^{3} kck^2 = \sum_{k=1}^{3} ck^3 = \frac{1}{14}(1+8+27) = \frac{18}{7}.$$

$$E[X^2] = \sum_{k=1}^{3} k^2ck^2 = \sum_{k=1}^{3} ck^4 = \frac{1}{14}(1+16+81) = 7, \quad \mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{19}{49}$$

2. (a) X can be modeled as a binomial random variable with parameters $(103, 0.95)$.
On average, $\mathsf{E}[X] = 103 \times 0.95 = 97.85$ passengers show up for the flight.

(b) $P\{X \leq 100\} = 1 - P\{X > 100\}$
$= 1 - P\{X = 101\} - P\{X = 102\} - P\{X = 103\}$
$= 1 - \binom{103}{101}(0.95)^{101}(0.05)^2 - \binom{103}{102}(0.95)^{102}(0.05)^1 - \binom{103}{103}(0.95)^{103}(0.05)^0$

(c) Y can be modeled as a binomial random variable with parameters $(103, 0.05)$.
On average, $\mathsf{E}[Y] = 103 \times 0.05 = 5.15$ passengers do not show up for the flight. Hence, $\lambda = E[Y] = 5.15$, and hence
$P\{Y \geq 3\} = 1 - P\{Y < 3\} = 1 - P\{Y = 2\} - P\{Y = 1\} - P\{Y = 0\}$
$\approx 1 - e^{-5.15}\frac{(5.15)^2}{2!} - e^{-5.15}\frac{(5.15)^1}{1!} - e^{-5.15}\frac{(5.15)^0}{0!} = 1 - e^{-5.15}\left(\frac{(5.15)^2}{2} + 5.15 + 1\right)$

3. (a) We use the fact there are 8 equally likely outcomes of the experiment: $\{HHH, HHT, HTH, \ldots, TTT\}$. The event A has 4 outcomes out of a total of 8, and therefore $P(A) = \frac{1}{2}$. Event B has 6 outcomes, and therefore $P(B) = \frac{3}{4}$. The event AB corresponds to "exactly one head" and has 3 outcomes, and so $P(AB) = \frac{3}{8}$. Since $P(AB) = P(A)P(B)$, A and B are independent.

(b) If A and B are independent and $B \subset A$, then $P(B) = P(AB) = P(A)P(B)$, from which we can conclude that either $P(B) = 0$ or $P(A) = 1$.

4. (a) Let $a = P\{A \text{ wins}\}$. Use the law of total probability based on what happens for the first flip. If it is heads, which happens with probability 1/2, A wins. If it is tails, which also happens with probability 1/2, then the conditional probability A wins is $1 - a$. (That is because if the first flip is tails, the remainder of the game is similar to the original game,

except with player B going first.) Thus, $a = \frac{1}{2} + \frac{1}{2}(1-a)$, or $\frac{3a}{2} = 1$ or $a = 2/3$. That is, $P\{A \text{ wins}\} = \frac{2}{3}$.
Alternatively, let L denote the total number of coin flips until heads appears. Then L has the geometric distribution with parameter 1/2, so $P\{L = k\} = 2^{-k}$ for $k \geq 1$. Then

$$\begin{aligned} P\{A \text{ wins}\} &= P\{L \text{ is odd}\} = 2^{-1} + 2^{-3} + 2^{-5} + \cdots \\ &= \frac{1}{2}\left(1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \cdots\right) = \frac{1}{2}\frac{1}{1-\frac{1}{4}} = \frac{2}{3} \end{aligned}$$

(b) There are several ways to solve the problem, but most follow one of the following approaches. Let L denote the total number of flips until heads appears.

$$\begin{aligned} P\{A \text{ wins}\} &= P\{L \in \{1,4,5,8,9,\ldots\}\} = 2^{-1} + 2^{-4} + 2^{-5} + 2^{-8} + 2^{-9} + \cdots \\ &= (2^{-1} + 2^{-5} + 2^{-9} + \cdots) + (2^{-4} + 2^{-8} + 2^{-12} + \cdots) \\ &= \left(\frac{1}{2} + \frac{1}{16}\right)\left(1 + \left(\frac{1}{16}\right) + \left(\frac{1}{16}\right)^2 + \cdots\right) \\ &= \frac{9}{16}\left(\frac{1}{1-\frac{1}{16}}\right) = \frac{9}{16}\frac{16}{15} = \frac{3}{5} \end{aligned}$$

or

$$\begin{aligned} P\{A \text{ wins}\} &= P\{L \in \{1,4,5,8,9,\ldots\}\} = 2^{-1} + 2^{-4} + 2^{-5} + 2^{-8} + 2^{-9} + \cdots \\ &= 2^{-1} + (1 + 2^{-1})\left(2^{-4} + 2^{-8} + 2^{-12} + \cdots\right) \\ &= \frac{1}{2} + \frac{3}{2}\left(\frac{1}{16}\right)\left(1 + \left(\frac{1}{16}\right) + \left(\frac{1}{16}\right)^2 + \cdots\right) \\ &= \frac{1}{2} + \frac{3}{2}\left(\frac{1}{16}\right)\left(\frac{1}{1-\frac{1}{16}}\right) = \frac{3}{5} \end{aligned}$$

Alternatively, let $a = P\{A \text{ wins}\}$. Use the law of total probability based on what happens for the first two flips. If the flips are either HH or HT then player A wins. If the flips are HT then player A loses. If the flips are TT then the game essentially starts over but with player B going first, so the conditional probability A wins in this case is $1-a$. Thus, $a = \frac{1}{4} + \frac{1}{4} + 0 + \frac{1}{4}(1-a)$. Solving for a yields $P\{A \text{ wins}\} = a = \frac{3}{5}$.

5. (a) Recall that the ML rule declares H_1 if $\Lambda(k) \geq 1$. In this case,

$$\Lambda(k) = \begin{cases} \frac{\left(\frac{1}{2}\right)^k}{\frac{1}{4}} = \frac{4}{2^k} & k \in \{2,3,4,5\} \\ \infty & \textit{else} \end{cases}$$

Clearly, we will declare H_1 if $k \notin \{2,3,4,5\}$. If $k \in \{2,3,4,5\}$, then $\Lambda(k) \geq 1$ only for $k = 2$. Hence, the ML rule is to declare H_0 if $k \in \{3,4,5\}$ and to declare H_1 if $k \in \{1,2,6,7,8,\ldots\}$.

(b) By definition, $p_{miss} = P\{\text{declare } H_0|H_1\} = P\{X \in \{3,4,5\}|H_1\} = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 = \frac{7}{32}$.

(c) The MAP rule is to decide H_1 if $\Lambda(k) \geq \frac{\pi_0}{\pi_1} = \frac{1}{3}$. From part (a), we can see that this is the case if $k \notin \{4,5\}$. Hence, the MAP rule is to decide H_1 if $X \notin \{4,5\}$ and decide H_0 if $X \in \{4,5\}$. Therefore, $p_{\text{false alarm}} = P\{\text{declare } H_1 | H_0\} = P\{X \notin \{4,5\} | H_0\} = P\{X \in \{2,3\} | H_0\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

6. (a) There are $\binom{6}{2}$ ways to choose 2 of your 6 school friends. For each one of those choices, there are $\binom{5}{2}$ ways to choose 2 of your 5 football friends, and for each one of those choices, there are $\binom{3}{2}$ ways to choose 2 of your 3 chess friends. Hence, the total number of such distinct teams is $\binom{6}{2}\binom{5}{2}\binom{3}{2} = 450$.

   (b) One way to do this is to realize that the total number of such distinct teams is the number of teams without any restrictions, given by $\binom{6+5+3}{6} = \binom{14}{6} = 3003$, minus the number of distinct teams missing members from at least one group. The number of teams without school friends is $\binom{5+3}{6} = \binom{8}{6} = 28$. The number of teams without football friends is $\binom{6+3}{6} = \binom{9}{6} = 84$. The number of teams without chess friends is $\binom{6+5}{6} = \binom{11}{6} = 462$. We then have to subtract the number of ways in which pairs of these absences occur (e.g. no school and no football), but the only way in which this can occur is if football and chess friends are missing (otherwise there are not enough friends to make the team). The number of teams without both football and chess friends is $\binom{6}{6} = 1$. Hence, the number of teams in this case is $3003 - (28 + 84 + 462 - 1) = 2430$.