**University of Illinois** Fall 2015

# ECE 313: Final Exam

Tuesday, December 15, 2015
7 p.m. — 10 p.m.
Room ECEB 1015

**Name: (in BLOCK CAPITALS)** ________________________________

**NetID:** ________________________________

**Signature:** ________________________________

**Section:**
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

## Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 10 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

## Grading

1. 12 points ____________
2. 18 points ____________
3. 22 points ____________
4. 16 points ____________
5. 24 points ____________
6. 8 points ____________
7. 28 points ____________
8. 22 points ____________
9. 20 points ____________
10. 30 points ____________

Total (200 points) ____________

1. [**12 points**] The three parts of this problem are unrelated.

   (a) Count the number of distinct 7-letter sequences that can be obtained from permutating the letters in the word "darling".

   (b) Count the number of distinct 6-letter sequences that can be obtained from permutating the letters in the word "banana".

   (c) There are 6 socks in a drawer: 2 black, 2 brown, and 2 gray. Suppose there are 3 siblings present, and one at a time, they each randomly grab two socks out of the drawer, without replacement.
   i) Find the probability that each sibling draws a matching pair of socks.
   ii) Find the probability that no sibling draws a matching pair of socks.

2. **[18 points]** Alice transmits a single bit $X$ to Bob over a noisy channel. For each transmission, the output is equal to $X$ with probability $p$ and $1 - X$ with probability $1 - p$, independently of other transmission attempts. Alice attempts $n$ transmissions and Bob, on the other end of the channel, observes $Y_1, \ldots, Y_n$, the $n$ outputs of the channel, and computes $S_n = Y_1 + \cdots + Y_n$.

   (a) Assuming that $X = 1$, find the pmf of $S_n$.

   (b) Assuming that $X = 1$, $E[S_n] = 12$ and $Var(S_n) = 3$, find the value of $p$ and $n$.

   (c) Assuming that $X = 1$. Find the maximum likelihood estimate of $n$ given that $S_n = 10$ is observed.

(d) Assuming that $X$ is a Bernoulli random variable with parameter $1/2$, find the pmf of $S_n$.

3. [**22 points**] Consider the experiment of rolling two fair dice. Define the two events $A = \{\text{both dice show a different number}\}$ and $B = \{\text{sum of numbers showing is} \leq 4\}$.

(a) Obtain $P(B|A)$ and determine if $A$ and $B$ are independent (explain).

(b) Repeat the experiment 10 times and let $X$ be the number of times that event $A$ occurs. Obtain the pmf of $X$.

(c) Let $Y = -3X$, with $X$ defined as above. Obtain $E[Y]$, $Var(Y)$ and $E[Y^2]$.

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(d) Are $X$ and $Y$, as defined above, independent? Explain.

4. **[16 points]** Let $X$ and $Y$ be jointly Gaussian random variables with conditional ditribution $f_{Y|X}(v|u) = \sqrt{\frac{1}{2\pi}} e^{-\frac{1}{2}(v-(5-u))^2}$.

   (a) Obtain the best linear estimator $\hat{E}[Y|X=2]$.

   (b) Obtain $E[Y|X=2]$.

   (c) Obtain $E[Y^2|X=2]$.

5. [**24 points**] Consider a random variable $X$ with pdf:

$$f_X(x;\theta) = \begin{cases} (1+\theta)x^\theta, & 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

We want to decide $H_1$ or $H_0$ upon the observation of $X$ given the following hypotheses:

$$\begin{aligned} H_0 &: \theta_0 = 2 \\ H_1 &: \theta_1 = 1 \end{aligned}$$

(a) Find the ML decision rule.

(b) Find the MAP decision rule if the priors $\pi_0 = P(H_0)$ and $\pi_1 = P(H_1)$ satisfy $3\pi_0 = 2\pi_1$.

(c) Compute $P_{\text{false alarm}}$ for both the ML and MAP decision rules obtained above.

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(d) Compute $P_{\text{miss detection}}$ for both the ML and MAP decision rules obtained above.

6. [**8 points**] Assume that a casino player has found a malfunctioning slot machine, which returns an average amount of \$40 every time the player plays. Additionally, the corresponding standard deviation of the amount the machine returns every time is \$5. In order to maximize his revenue, the player has constructed a number of fake slot machine coins so that he can play at no cost. If he plays for 100 times, with what probability will he win at least \$3900, assuming the trials are independent and identically distrbuted.? [Hint: Use Central Limit Theorem]

7. **[28 points]** Emails arrive at Jim's inbox according to a Poisson process with rate $\lambda = 1$ per hour.

   (a) Find the probability that Jim receives at least 3 emails between noon and 2pm.

   (b) Let $T_1$ denote the arrival time of the first email. Find the pdf of $T_1$.

   (c) Find the mean and variance of $T_1$.

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(d) Let $T_2$ denote the arrival time of the second email. Compute the correlation coefficient between $T_1$ and $T_2$.

(e) Suppose each email has a chance $\frac{1}{2}$ of being a spam email independently of everything else. Find the probability that Jim receives no spam from noon to 2pm.

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(f) Let $S$ denote the arrival time of the first spam. Find its CDF and pdf.

8. **[22 points]** Alice and Bob plan to go to lunch. They each arrive at a random time between noon and 1pm independently of each other. Denote the arrival time of Alice and Bob by $X$ and $Y$ respectively, which are both uniformly distributed over $[0, 1]$ and mutually independent.

   (a) Find the probability that Alice arrives earlier than Bob.

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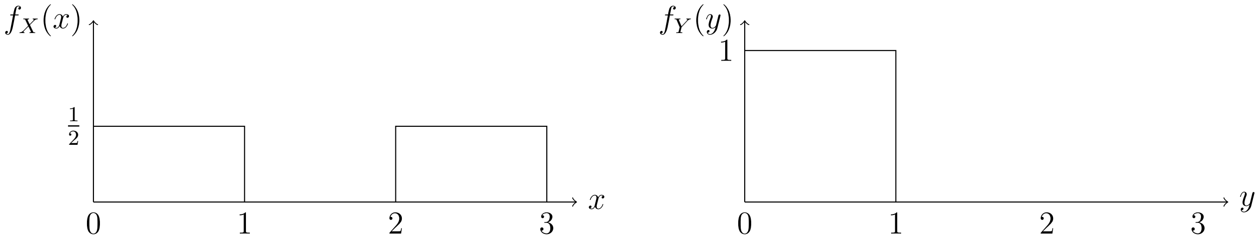
(b) Let $Z$ denote the arrival time of the earliest of the two. Find the pdf of $Z$ and the mean $E[Z]$.

(c) Let $W$ denote the arrival time of the latest of the two. Find its mean $E[W]$.

(d) Suppose both Alice and Bob are willing to wait for each other for at most 30 minutes, that is, whoever arrives the first will leave if the other person does not show up in 30 minutes. Find the probability that Alice and Bob meet.

9. **[20 points]** Let $X$ and $Y$ be independent random variables with pdfs plotted below:




(a) Find the correlation $E[XY]$ and the correlation coefficient $\rho_{X,Y}$.

(b) Let $S = X + Y$. Sketch the pdf $f_S$, clearly labelling all important points.

(c) Let $T = X - Y$. Sketch the pdf $f_T$, clearly labelling all important points.

10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Consider a binary hypothesis testing problem with $H_0 : X \sim \mathcal{N}(0, 1)$ and $H_1 : X \sim \mathcal{N}(1, 1)$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | If ML rule is employed, then $P_{\text{false alarm}} = P_{\text{miss detection}}$. |
| ☐ | ☐ | If MAP rule is employed, then $P_{\text{false alarm}}$ and $P_{\text{miss detection}}$ must be different. |
| ☐ | ☐ | $P_e = P_{\text{false alarm}} P(H_1) + P_{\text{miss detection}} P(H_0)$ always holds. |

(b) Let $A, B$ be two events in the sample space $\Omega$ with non-zero probabilities. Let $E_1, E_2, \ldots, E_n$ be a partition of $\Omega$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | If $A, B$ are mutually exclusive, then $A, B$ must be independent. |
| ☐ | ☐ | $\sum_{i=1}^{n} P(A\|E_i) = \sum_{i=1}^{n} P(B\|E_i)$ always holds. |
| ☐ | ☐ | $\sum_{i=1}^{n} P(E_i\|A) = \sum_{i=1}^{n} P(E_i\|B)$ always holds. |

(c) Assume that $X_1, X_2, \ldots, X_n$ are identically distributed random variables with mean $-\infty < \mu < +\infty$ and variance $0 < \sigma^2 < +\infty$. Let $S_n = X_1 + \ldots + X_n$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | If $X_1, X_2, \ldots, X_n$ are also independent, then for any $\epsilon > 0$, $P\left(\left\|\frac{S_n}{n} - \mu\right\| \geq \epsilon\right) \to 0$ as $n \to +\infty$. |
| ☐ | ☐ | If $X_1 = X_2 = \cdots = X_n$, then for any $\epsilon > 0$, $P\left(\left\|\frac{S_n}{n} - \mu\right\| \geq \epsilon\right) \to 0$ as $n \to +\infty$. |

(d) Flip a biased coin which results in a head with probability 0.55. Define the random variables $X$ and $Y$ as follows: If the outcome is head, set $X = 1$ and $Y = 0$. If the outcome is tail, set $X = 0$ and $Y = 1$. Cross the box to the left of the correct option.

i) The correlation coefficient is $\rho_{X,Y}$ is ☐ 0 ☐ 1 ☐ 0.55 ☐ − 1.

ii) The MMSE of estimating $Y$ based on $X$ is: ☐ 1 ☐ 2 ☐ 0.55 ☐ 0 .