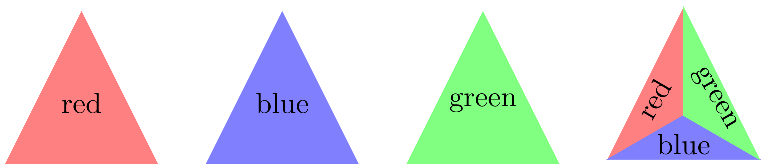
University of Illinois Fall 2015

ECE 313: Hour Exam I

Wednesday, October 7, 2015
7:00 p.m. — 8:15 p.m.
A-J in MSEB 100, K-R in DCL 1320, S-Z in EVRT 151

1. **[14 points]** A tetrahedron has four faces, which are painted as follows: one side all red, one side all blue, one side all green, and one side with red, blue and green.



Toss the tetrahedron randomly and the face that lands on the floor is equally likely among the four. Define the following events:

$$R = \{\text{the face that hits the floor has red color}\}$$
$$B = \{\text{the face that hits the floor has blue color}\}$$
$$G = \{\text{the face that hits the floor has green color}\}$$

(a) Compute the probabilities: $P[R], P[G], P[B]$.
Solution: $P[R] = P[G] = P[B] = 1/2$.

(b) Are the events R, G, B pairwise independent? Justify your answer by calculations.
Solution: $P[RG] = P[GB] = P[BR] = P[\text{4th face}] = 1/4$. So they are pairwise independent.

(c) Are the events R, G, B independent? Justify your answer by calculations.
Solution: $P[RGB] = P[\text{4th face}] = 1/4 \neq P[R]P[G]P[B]$. So they are not independent.

2. **[15 points]** In a classroom there are n students whose birthday are equally likely chosen from k different dates.

(a) One of the students is named Bob. Let X denote the number of other students who were born on the same day as Bob. What is the distribution of X?
Solution: Binomial$(n-1, 1/k)$.

(b) What is the probability that there is no other student who was born on the same day as Bob?
Solution: $(1-1/k)^{n-1}$.

(c) What is the probability that there is no pair of students in the classroom who were born on the same day? Consider the cases $n \le k$ and $n > k$ separately.
Solution: If $n > k$, then there must exist a pair of students who are born on the same day so the probability in question is one; if $n \le k$, then $P[\text{no same birthday}] = \frac{k(k-1)\cdots(k-n+1)}{k^n} = (1-\frac{1}{k})(1-\frac{2}{k})\cdots(1-\frac{n-1}{k})$.

3. **[15 points]** The number of telephone calls arriving at a switchboard during any 10-minute period is known to be a Poisson random variable with $\lambda = 2$.

(a) What is the expected number of calls that will arrive during a 10-minute period?
Solution: Let X be the number of phone calls arriving at a switchboard during any 10-minute period. X is Poisson distributed with parameter $\lambda = 2$. The expected number of calls in any 10-minute period is $\mathbb{E}[X] = \lambda = 2$.

(b) Find the probability that more than three calls will arrive during a 10-minute period.
Solution: $P(X > 3) = 1 - P(X \le 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) = 1 - e^{-2}(1 + 2 + 2 + \frac{4}{3}) = 1 - \frac{19}{3}e^{-2}$.

(c) Find the probability that no calls will arrive during a 10-minute period.
Solution: $P(X = 0) = e^{-2}$.

4. **[18 points]** Roll a fair die and let D be the number showing. Then, choose D cards at random (without replacement) from a standard deck of 52 cards and let K be the number of kings that you get.

(a) Obtain $P\{K = 3|D = 4\}$.
Solution: Given that $D = 4$ there are $\binom{4}{3}$ ways to choose 3 kings out of the 4 kings, and $\binom{48}{1}$ ways to choose the remaining card out of the 48 that are not kings. Therefore,

$$P\{K = 3|D = 4\} = \frac{\binom{4}{3}\binom{48}{1}}{\binom{52}{4}}.$$

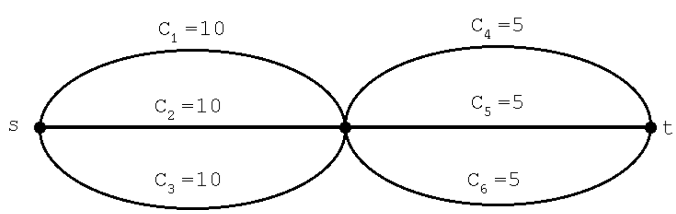
(b) Obtain $P\{K = 3\}$.
Solution: Using total probability

$$\begin{aligned} P\{K = 3\} &= \sum_{i=1}^{6} P\{K = 3|D = i\}P\{D = i\} = \sum_{i=3}^{6} P\{K = 3|D = i\}P\{D = i\} \\ &= \frac{\binom{4}{3}}{\binom{52}{3}}\left(\frac{1}{6}\right) + \frac{\binom{4}{3}\binom{48}{1}}{\binom{52}{4}}\left(\frac{1}{6}\right) + \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}\left(\frac{1}{6}\right) + \frac{\binom{4}{3}\binom{48}{3}}{\binom{52}{6}}\left(\frac{1}{6}\right) \end{aligned}$$

(c) Obtain $P\{D = 4|K = 3\}$.
Solution: Using Bayes rule

$$P\{D = 4|K = 3\} = \frac{P\{K = 3|D = 4\}P\{D = 4\}}{P\{K = 3\}} = \frac{\frac{\binom{4}{3}\binom{48}{1}}{\binom{52}{4}}\left(\frac{1}{6}\right)}{\frac{\binom{4}{3}}{\binom{52}{3}}\left(\frac{1}{6}\right) + \frac{\binom{4}{3}\binom{48}{1}}{\binom{52}{4}}\left(\frac{1}{6}\right) + \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}\left(\frac{1}{6}\right) + \frac{\binom{4}{3}\binom{48}{3}}{\binom{52}{6}}\left(\frac{1}{6}\right)}$$

5. **[18 points]** Consider the following $s - t$ flow network, where link i has the indicated capacity C_i, and link i fails with probability p_i independently of other links.



(a) What possibles values of capacity can be achieved in this $s - t$ flow network?
Solution: $C \in \{0, 5, 10, 15\}$.

(b) Obtain the pmf of the capacity of this $s-t$ flow network.
Solution: Let $q_i = 1 - p_i$ and $F_i = \{\text{link } i \text{fails}\}$ then

$$\begin{aligned}
p_C(0) &= P\{\text{all left links fail OR all right links fail}\} = P\{F_1F_2F_3 \cup F_4F_5F_6\} \\
&= p_1p_2p_3 + p_4p_5p_6 - p_1p_2p_3p_4p_5p_6 \\
p_C(5) &= P\{\text{only one right link works AND at least one left link works}\} \\
&= P\{(F_4F_5F_6^c \cup F_4F_5^cF_6 \cup F_4^cF_5F_6)(F_1F_2F_3)^c\} \\
&= (p_4p_5q_6 + p_4q_5p_6 + q_4p_5p_6)(1 - p_1p_2p_3) \\
p_C(10) &= P\left\{ \begin{array}{l} \{\text{only two right links work AND at least one left link works}\} \\ OR \ \{\text{all right links work AND only one left link works}\} \end{array} \right\} \\
&= P\left\{ \begin{array}{l} \{(F_4F_5^cF_6^c \cup F_4^cF_5^cF_6 \cup F_4^cF_5F_6^c)(F_1F_2F_3)^c\} \\ \cup \ \{F_4^cF_5^cF_6^c(F_1F_2F_3^c \cup F_1F_2^cF_3 \cup F_1^cF_2F_3)\} \end{array} \right\} \\
&= (p_4q_5q_6 + q_4q_5p_6 + q_4p_5q_6)(1 - p_1p_2p_3) + q_4q_5q_6(p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3) \\
p_C(15) &= P\{\text{all right links work AND at least two left links work}\} \\
&= P\{F_4^cF_5^cF_6^c(F_1F_2^cF_3^c \cup F_1^cF_2^cF_3 \cup F_1^cF_2F_3^c \cup F_1^cF_2^cF_3^c)\} \\
&= q_4q_5q_6(p_1q_2q_3 + q_1q_2p_3 + q_1p_2q_3 + q_1q_2q_3)
\end{aligned}$$

(c) Use the union bound to bound the probability of outage of this network.
Solution:

$$p_C(0) = P\{F_1F_2F_3 \cup F_4F_5F_6\} \leq P\{F_1F_2F_3\} + P\{F_4F_5F_6\} = p_1p_2p_3 + p_4p_5p_6$$

6. **[20 points]** Consider a random variable X uniformly distributed on the set $\{1, \ldots, n\} \cup \{2n+1, \ldots, 3n\}$, i.e. $P\{X = k\}$ is constant for $k = 1, \ldots, n, 2n+1, \ldots, 3n$.

(a) Suppose that n is unknown but it is observed that $X = 9$. Obtain the maximum likelihood estimate of n.
Solution: $P_X(k) = \frac{1}{2n}$ whenever $k \in 1, \ldots, n, 2n+1, \ldots, 3n$, $P_X(k) = 0$ otherwise. Therefore, we would like to find the smallest n that is consistent with our observation $X = 9$. This is achieved when $n = 3$.

(b) Suppose now that it is known that n can have two different known values, n_1 and n_2, which gives rise to two hypotheses

$$H_0 : X \in \{1, ..., n_1\} \cup \{2n_1 + 1, ...3n_1\},$$

$$H_1 : X \in \{1, ..., n_2\} \cup \{2n_2 + 1, ..., 3n_2\},$$

where $n_1 < n_2 < 2n_1$. Obtain the maximum likelihood decision rule.
Solution: $P(X|H_0) = \frac{1}{2n_1}$ whenever $X \in \{1, ..., n_1\} \cup \{2n_1 + 1, ...3n_1\}$, $P(X|H_0) = 0$ otherwise. Similarly, $P(X|H_1) = \frac{1}{2n_2}$ whenever $X \in \{1, ..., n_2\} \cup \{2n_2 + 1, ..., 3n_2\}$, $P(X|H_1) = 0$ otherwise. Since $n_1 < n_2$, $P(X|H_0) > P(X|H_1)$ whenever $X \in \{1, ..., n_1\} \cup \{2n_1 + 1, ...3n_1\}$. Therefore, the maximum likelihood rule will choose H_0 whenever $X \in \{1, ..., n_1\} \cup \{2n_1 + 1, ...3n_1\}$, H_1 otherwise.

(c) Using the decision rule and distributions from part (b), obtain $p_{\text{false alarm}}$.
Solution: $p_{\text{false alarm}} = P(\text{say } H_1 | H_0 \text{ is true}) = 0$ because the maximum likelihood decision rule will always say H_0 when H_0 is true.