University of Illinois ECE 313: Final Exam Fall 2014

Monday, December 15, 2014, 7:00 p.m. — 10:00 p.m.

Sect. B, names A-O, 1013 ECE, names P-Z, 1015 ECE; Section C, names A-L, 1015 ECE; all others 112 Gregory

1. (a) On the average, it takes 6 rolls until the three comes up the first time and six more until the three comes up again, so $E[N] = 12$. We could also note that N has the negative binomial distribution with parameters $r = 2$ and $p = \frac{1}{6}$.

   (b) We need to identify the pmf of N. For $k \geq 2$ the event $N = k$ happens if exactly one of the first $k-1$ rolls is a three and the k^{th} roll is also a three, so $P\{N = k\} = \left(\binom{k-1}{1}(1-p)^{k-2}p\right)p = (k-1)(1-p)^{k-1}p^2$, where $p = \frac{1}{6}$. Using LOTUS and the formula for the sum of a geometric series,

$$\begin{aligned} E\left[\frac{1}{N-1}\right] &= \sum_{k=2}^{\infty} \frac{1}{k-1}(k-1)(1-p)^{k-2}p^2 \\ &= p^2 \sum_{k=2}^{\infty}(1-p)^{k-2} = p^2\left(\frac{1}{1-(1-p)}\right) = p = \frac{1}{6}. \end{aligned}$$

2. (a) The largest value that Y can take with non-zero probability is 30, and it takes this value if and only none of the links fail. Therefore

$$P\{Y \geq 30\} = P\{Y = 30\} = P(\cap_{i=1}^{6} F_i^c) = (1-p)^6 = q^6.$$

   (b) For Y to take the value 20, either link 5 fails and all the other links are okay, or links 1, 4, 5 and 6 are okay and either of links 2 or 3 fail. Therefore

$$\begin{aligned} P\{Y = 20\} &= P(F_5)P(F_1^c F_2^c F_3^c F_4^c F_6^c) + P(F_1^c F_4^c F_5^c F_6^c)P(F_2 \cup F_3) \\ &= pq^5 + q^4(2p - p^2) \end{aligned}$$

3. (a) There are 4^8 possible words that M_4 can produce, out of which 2^8 are bytes and therefore the probability that M_4 produces a byte is 2^{-8}.

   (b)

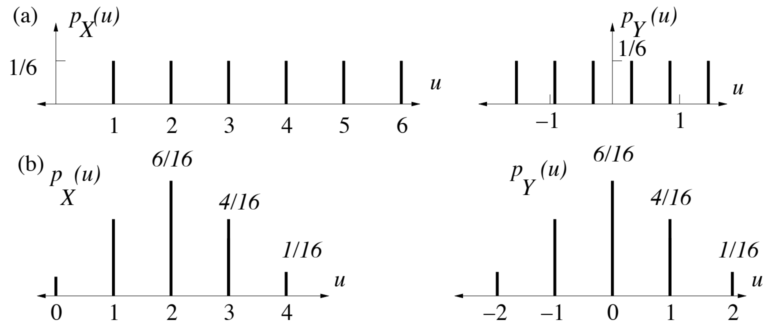
$$P(\text{byte}) = P(\text{byte}|M_2)P(M_2) + P(\text{byte}|M_4)P(M_4) = (1)\frac{1}{2} + \frac{1}{2}\frac{1}{2^8} = 2^{-1} + 2^{-9}$$

   (c)

$$P(M_2|\text{byte}) = \frac{P(\text{byte}|M_2)P(M_2)}{P(\text{byte})} = \frac{2^{-1}}{2^{-1} + 2^{-9}} = \frac{2^8}{1 + 2^8}$$

   (d) X has a binomial$(10, p)$ pmf with $p = 2^{-8}$, so $E[X] = 10p$ and $\text{Var}(X) = 10p(1-p)$.

4. (a) The mean of X is 3.5, so to get the pmf of Y we shift the pmf of X to the left by 3.5 (i.e. we center it) and then scale by shrinking the pmf horizontally by the factor 1.7. The values of the pmf for Y are still all 1/6.



(b) The mean of X is $np = 2$ and the variance is $np(1-p) = 1$. Since the variance is already one, $Y = X - 2$; no scaling is necessary. The pmf of Y is the pmf of X shifted to the left by two (i.e. it is centered).

5. (a) By the memoryless property of the exponential distribution,
$P(T \leq 12|T \geq 5) = P\{T \leq 7\} = 1 - e^{-7\lambda}$.
ALTERNATIVELY,
$P(T \leq 12|T \geq 5) = \frac{P\{5 \leq T \leq 12\}}{P\{T \geq 5\}} = \frac{P\{X \geq 5\} - P\{X \geq 12\}}{P\{X \geq 5\}} = \frac{e^{-5\lambda} - e^{-12\lambda}}{e^{-5\lambda}} = 1 - e^{-7\lambda}$.

(b) (Since $x^2 + xT + 1 > 0$ for $x = 0$ the event $\{x^2 + xT + 1 \geq 0$ for all $x \in \mathbb{R}\}$ is equivalent to the discriminant, $T^2 - 4$, of the quadratic expression being less than or equal to zero (i.e. the quadratic equation $x^2 + xT + 1 = 0$ not having distinct real roots), or equivalently, $\{T \leq 2\}$, which has probability $1 - e^{-2\lambda}$.

6. (a) Expanding, and using the independence of X and Y (so $\mathrm{Cov}(X, Y) = 0$) yields
$\mathrm{Cov}(X, S + Y) = \mathrm{Cov}(X, X + 2Y) = \mathrm{Cov}(X, X) + 2\mathrm{Cov}(X, Y) = \mathrm{Var}(X) = p(1 - p)$.

(b) The sum of n independent Bernoulli random variables with the same parameter has the binomial distribution with parameters n and p. Hence, S is binomial with parameters n and p.

(c) From the convolution formula for independent random variables, or equivalently, the law of total probability,
$P\{S = 3\} = P\{X = 0\}P\{Y = 3\} + P\{X = 1\}P\{Y = 2\} = (1 - p)q(1 - q)^2 + pq(1 - q)$.

7. (a) The joint density is easily recognized to be the product of two densities of exponential type with parameter one. (This observation is not necessary but it simplifies the rest of this part.) Let $W = X^2$. Clearly W is a nonnegative random variable. The pdf of W can be computed from the formula $f_W(v) = f_X(u)/g'(u)$, where $v = g(u) = u^2$ and $u > 0$. Hence, $u = \sqrt{v}$ so $f_W(v) = \exp(-\sqrt{v})/(2\sqrt{v})$, for $v > 0$, and zero otherwise.
ALTERNATIVELY, we can find the CDF of W and differentiate it. For $c \geq 0$, $F_W(c) = P\{Y \leq \sqrt{c}\} = 1 - \exp(-\sqrt{c})$. Differentiating gives the same answer as before.

(b)

$$\begin{aligned} P\{X + Y \leq 1\} &= \int_0^1 \int_0^{1-u} e^{-u-v} dv du = \int_0^1 e^{-u}(1 - e^{-(1-u)}) du \\ &= \int_0^1 e^{-u} - e^{-1} du = 1 - 2e^{-1}. \end{aligned}$$

ALTERNATIVELY, $X + Y$ can be viewed as the time of the second count of a Poisson process with rate one, and $\{X + Y \leq 1\}$ is the same as the event $N_1 \geq 2$, which has probability $1 - P\{N_1 = 0\} - P\{N_1 = 1\} = 1 - 2e^{-1}$.

(c) This corresponds to applying the mapping $\alpha = u^2$ and $\beta = u + 2v$ from the positive quadrant of the $u - v$ plane to the $\alpha - \beta$ plane. The support of $f_{W,Z}$ is the image set, $\alpha \geq 0$ and $\beta \geq \sqrt{\alpha}$. The mapping is one-to-one because u, v can be found from α and β, namely $u = \sqrt{\alpha}$ and $v = (\beta - \sqrt{\alpha})/2$. The Jacobian of the mapping is $4u = 4\sqrt{\alpha}$, Therefore,

$$f_{W,Z}(\alpha, \beta) = \begin{cases} \frac{1}{4\sqrt{\alpha}} e^{-\sqrt{\alpha} - (\beta - \sqrt{\alpha})/2} & \alpha > 0, \beta \geq \sqrt{\alpha} \\ 0 & \text{else} \end{cases}$$

8. First, we know that N_3 and N_1 are Poisson random variables with parameters 6 and 2, respectively. So the means are: $E[N_3] = 6$ and $E[N_1] = 2$, and the variances are: $\mathrm{Var}(N_3) = 6$ and $\mathrm{Var}(N_1) = 2$. To compute the covariance, we use the fact $N_3 = N_1 + (N_3 - N_1)$, and the fact N_1 and $N_3 - N_1$ are mutually independent, so $\mathrm{Cov}(N_1, N_3 - N_1) = 0$, yielding

$$\mathrm{Cov}(N_1, N_3) = \mathrm{Cov}(N_1, N_1) + \mathrm{Cov}(N_1, N_3 - N_1) = \mathrm{Var}(N_1) = 2.$$

Finally, the formula for the LMMSE yields

$$\widehat{N_1} = \widehat{E}[N_1 | N_3 = 5] = E[N_1] + \frac{\mathrm{Cov}(N_1, N_3)}{\mathrm{Var}(N_3)}(5 - E[N_3] = 2 + \frac{2}{6}(5 - 6) = \frac{5}{3}.$$

ALTERNATIVELY, we have seen in Example 3.5.3(d) that given the number of points arriving up to some time t, the times of arrival are independent and uniformly distributed. Thus, given $N_3 = n$, the conditional distribution of N_1 is binomial with parameters n and $p = \frac{1}{3}$. Thus, $E[N_1 | N_3 = n] = \frac{n}{3}$ for all $n \geq 0$. Since this best unconstrained estimator is linear, it is equal to the MMSE linear estimator as well. That is, $\widehat{E}[N_1 | N_3 = n] = \frac{n}{3}$. Setting $n = 5$ gives the same answer as before.

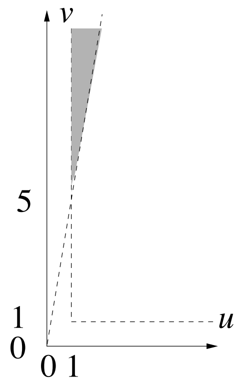
9. (a) By integrating out v, you can find that $f_X(u) = \begin{cases} \frac{1}{u^2} & u \geq 1 \\ 0 & \text{else} \end{cases}$. More simply, you can identify f_X by noticing that the joint pdf factors, and f_X is the same as f_Y. In fact, X and Y are independent. Therefore, $f_{Y|X}(v|5)$ is the same as $f_Y(v)$:

$$f_{Y|X}(v|5) = \begin{cases} \frac{1}{v^2} & v \geq 1 \\ 0 & \text{else} \end{cases}.$$

(b) By LOTUS,

$$E\left[\frac{1}{X}\right] = \int_1^\infty \int_1^\infty \frac{1}{u} \frac{1}{u^2 v^2} du dv = \int_1^\infty \frac{1}{u^3} du \int_1^\infty \frac{1}{v^2} dv = \frac{1}{2}.$$

(c) Integrate over the intersection of the support of the pdf and the region $\{v \geq 5u\}$, shown as the shaded region, to get:



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\begin{aligned}
P\{Y \geq 5X\} &= \int_1^\infty \int_{5u}^\infty \frac{1}{u^2 v^2} dv du \\
&= \int_1^\infty \frac{1}{u^2} \left(-\frac{1}{v}\right)\Big|_{5u}^\infty du \\
&= \int_1^\infty \frac{1}{5u^3} du = \frac{1}{10}.
\end{aligned}
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10. (a) Let $c_{i,j} = \text{Cov}(X_i, X_j)$. Then
    $\text{Cov}(X_1 + X_2, X_1 + X_3) = c_{1,1} + c_{1,3} + c_{2,1} + c_{2,3} = 5 + 0 + 2 + 2 = 9$.

    (b) $\text{Cov}(X_2 - aX_1, X_1) = c_{2,1} - ac_{1,1} = 2 - 5a$, which is zero for $a = \frac{2}{5}$.

    (c) $\text{Var}(X_i) = \text{Cov}(X_i, X_i) = 5$ for all i and $\text{Cov}(X_1, X_2) = 2$.
    So $\rho_{X_1,X_2} = \frac{2}{\sqrt{5 \cdot 5}} = \frac{2}{5}$.

    (d) $\text{Var}(X_1 + X_2 + X_3) = \text{Cov}(\sum_{i=1}^3 X_i, \sum_{j=1}^3 X_j) = 23$, because the covariance expands out to the sum of all entries in the covariance matrix.

11. (a) Note that $S = X_1 + \ldots + X_{25}$ where X_i denotes the weight of the i^{th} passenger. Then $E[S_{25}] = 25(150) = 3750$ and the standard deviation of S is $\sqrt{25(30)^2} = \sqrt{25}(30) = 150$.

    (b) We apply the Gaussian approximation suggested by the central limit theorem to get: $P\{S \geq 4000\} = P\left\{\frac{S-3750}{150} \geq \frac{250}{150}\right\} \approx Q(\frac{250}{150}) = Q(1.666) = 1 - \Phi(1.666) = 1 - 0.9522 = 0.0478$.

12. (a) False, True, False

    (b) True,True

    (c) True,False

    (d) True, True,True