University of Illinois ECE 313: Conflict Final Exam Fall 2014

Tuesday, December 16, 2014, 7:00 p.m. — 10:00 p.m. Room 2013 ECE Building

1. (a) Let c denote the fraction of nonsmoking women who get lung cancer (over some time period). Then $13c$ is the fraction of smoking women who get lung cancer over the same time period. The probability a typical woman gets lung cancer is the sum of the probability the women is a smoker and gets lung cancer plus the probability she is a nonsmoker and gets lung cancer, or

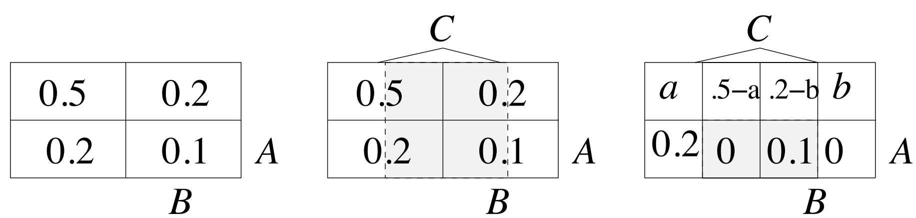
$$(0.15)13c + (0.85)c.$$

Thus, the conditional probability a women is a smoker given she got lung cancer is the first of the terms over the sum:

$$\frac{(0.15)13c}{(0.15)13c + (0.85)c} = \frac{(0.15)13}{(0.15)13 + (0.85)} = \frac{1.95}{2.80} \approx 67\%.$$

(b) The fraction of adults that smoke, namely 18%, is the average of the fraction of women that smoke, 15%, and the fraction of men that smoke. It follows that 21% of men smoke because $\frac{15+21}{2} = 18$. Thus, the ratio of the number of women that smoke to the total number of adults that smoke is 15/(15+21)=15/36=5/12.

2. (a) Using $P(A) = P(B) = 1$ and $P(A^c|B) = \frac{2}{3}$ or $P(A^cB) = \frac{2}{3}P(B) = 0.2$ allows us to easily find the Karnaugh map; see the first map shown below.



(b) To solve this problem we imagine overlaying event C on the Karnaugh map for A and B already found. Event C is shaded gray in the second Karnaugh map pictured. To specify a Karnaugh map for the three events, the probabilities for each of the four sets AB, AB^c, A^cB, A^cB^c have to be split between the parts of those sets inside C and outside C. The given condition $P(ABC) = 0.1$ determines $P(ABC^c) = 0$. To minimize $P(AC)$, shown in gray in the third map pictured, we put all of the probability for $P(AB^c)$ outside of C. That gives $P(AC) = 0.1$, the minimum possible. (Another way to see no smaller value of $P(AC)$ is possible is that $P(AC) \geq P(ABC) = 0.1$) All possible correct answers are shown in the third map pictured, were $a, b \geq 0$ with $a + b = 0.2$.

3. (a) T is the *minimum* of two independent exponential random variables, both with parameter λ. As seen in this course, the minimum is *also* an exponential random variable with parameter 2λ. So the pdf of T is $f_T(u) = 2\lambda e^{-2\lambda u}$ for $u \geq 0$ and zero otherwise.

(b) The solution is $\frac{1}{2\lambda}$ because that is the mean of an exponentially distributed random variable with parameter 2λ.

4. (a) There are 8^2 squares of side length 1, 7^2 squares of side length 2, etc., down to one square of side length 8. So the total number of squares is 64+49+36+25+16+9+4+1=204.

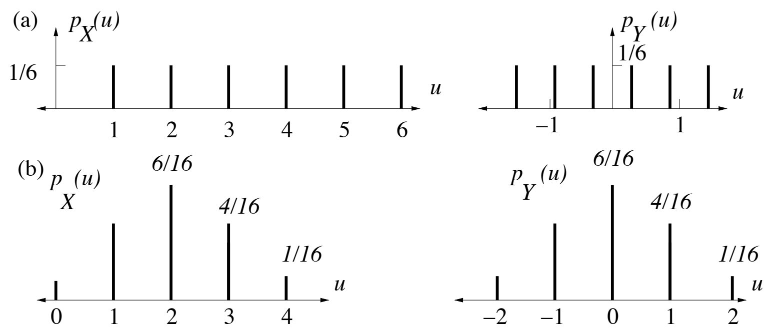
Another way to say it is that there are $(9-k)^2$ squares of side length k, for $1 \leq k \leq 8$). So $p_X(k) = \frac{(9-k)^2}{204}$ for $1 \leq k \leq 8$, and other values of p_X are zero.

(b) By LOTUS, $E\left[\frac{1}{9-X}\right] = \sum_{k=1}^{8} \frac{1}{9-k} \frac{(9-k)^2}{204} = \sum_{k=1}^{8} \frac{9-k}{204} = \frac{36}{204} = \frac{3}{17}$.

5. (a) The probability a serial network with n links fails is $1-(1-p)^n$, so we have $1-(1-p)^2 = 1-(1/2)^4$. Equivalently, $(1-p)^2 = (1/2)^4$, or $1-p = (1/2)^2$, or $p = 3/4$. (This makes sense. It makes the probability the first link of $\mathcal{N}$ is working the same as the probability the first two links of $\mathcal{M}$ are working.)

(b) The probability of simultaneous failure is $p(1-(1-p)^3) = \frac{1}{2}\frac{7}{8} = 7/16$ for the 3+1 design and $(1-(1-p)^2)^2 = (3/4)^2 = 9/16$ for the 2+2 design. Hence, the 3+1 design has the smaller probability of simultaneous failure.

6. (a) The mean of X is 3.5, so to get the pmf of Y we shift the pmf of X to the left by 3.5 (i.e. we center it) and then scale by shrinking the pmf horizontally by the factor 1.7. The values of the pmf for Y are still all 1/6.



(b) The mean of X is $np = 2$ and the variance is $np(1-p) = 1$. Since the variance is already one, $Y = X - 2$; no scaling is necessary. The pmf of Y is the pmf of X shifted to the left by two (i.e. it is centered).

7. (a) Expanding, and using the independence of X and Y (so $\mathrm{Cov}(X,Y) = 0$) yields $\mathrm{Cov}(X, S+Y) = \mathrm{Cov}(X, X+2Y) = \mathrm{Cov}(X,X) + 2\mathrm{Cov}(X,Y) = \mathrm{Var}(X) = p(1-p)$.

(b) The sum of n independent Bernoulli random variables with the same parameter has the binomial distribution with parameters n and p. Hence, S is binomial with parameters n and p.

(c) From the convolution formula for independent random variables, or equivalently, the law of total probability,
$P\{S=3\} = P\{X=0\}P\{Y=3\} + P\{X=1\}P\{Y=2\} = (1-p)q(1-q)^2 + pq(1-q)$.

8. (a) The joint density is easily recognized to be the product of two densities of exponential type with parameter one. (This observation is not necessary but it simplifies the rest of this part.) Let $W = X^2$. Clearly W is a nonnegative random variable. The pdf of W can be computed from the formula $f_W(v) = f_X(u)/g'(u)$, where $v = g(u) = u^2$ and $u > 0$. Hence, $u = \sqrt{v}$ so $f_W(v) = \exp(-\sqrt{v})/(2\sqrt{v})$, for $v > 0$, and zero otherwise. ALTERNATIVELY, we can find the CDF of W and differentiate it. For $c \geq 0$, $F_W(c) = P\{Y \leq \sqrt{c}\} = 1 - \exp(-\sqrt{c})$. Differentiating gives the same answer as before.

(b)

$$\begin{aligned} P\{X+Y \leq 1\} &= \int_0^1 \int_0^{1-u} e^{-u-v} dv du = \int_0^1 e^{-u}(1-e^{-(1-u)}) du \\ &= \int_0^1 e^{-u} - e^{-1} du = 1 - 2e^{-1}. \end{aligned}$$

ALTERNATIVELY, $X+Y$ can be viewed as the time of the second count of a Poisson process with rate one, and $\{X+Y \leq 1\}$ is the same as the event $N_1 \geq 2$, which has probability $1 - P\{N_1 = 0\} - P\{N_1 = 1\} = 1 - 2e^{-1}$.

(c) This corresponds to applying the mapping $\alpha = u^2$ and $\beta = u + 2v$ from the positive quadrant of the $u-v$ plane to the $\alpha - \beta$ plane. The support of $f_{W,Z}$ is the image set, $\alpha \geq 0$ and $\beta \geq \sqrt{\alpha}$. The mapping is one-to-one because u, v can be found from α and β, namely $u = \sqrt{\alpha}$ and $v = (\beta - \sqrt{\alpha})/2$. The Jacobian of the mapping is $4u = 4\sqrt{\alpha}$, Therefore,

$$f_{W,Z}(\alpha, \beta) = \begin{cases} \frac{1}{4\sqrt{\alpha}} e^{-\sqrt{\alpha}-(\beta-\sqrt{\alpha})/2} & \alpha > 0, \beta \geq \sqrt{\alpha} \\ 0 & \text{else} \end{cases}$$

9. First, we know that N_3 and N_1 are Poisson random variables with parameters 6 and 2, respectively. We find that the means are: $E[N_3] = 6$ and $E[N_1] = 2$. Further, the variances are: $\mathrm{Var}(N_3) = 6$ and $\mathrm{Var}(N_1) = 2$. Second, we find (as in midterm two) $E[N_3 N_1] = 14$. So the covariance between N_3 and N_1 is: $\mathrm{Cov}(N_3, N_1) = 14 - 12 = 2$. We see that the correlation coefficient ρ between N_3 and N_1 is $1/\sqrt{3}$.

   Using the formula for the LMMSE we get

$$\widehat{N_1} = \widehat{E}[N_1 | N_3 = 5] = 2 + \frac{2}{6}(5-6) = \frac{5}{3}.$$

   ALTERNATIVELY, we have seen in Example 3.5.3(d) that given the number of points arriving up to some time t, the times of arrival are independent and uniformly distributed. Thus, given $N_3 = n$, the conditional distribution of N_1 is binomial with parameters n and $p = \frac{1}{3}$. Thus, $E[N_1 | N_3 = n] = \frac{n}{3}$ for all $n \geq 0$. Since this best unconstrained estimator is linear, it is equal to the MMSE linear estimator as well. That is, $\widehat{E}[N_1 | N_3 = n] = \frac{n}{3}$. Setting $n = 5$ gives the same answer as before.

10. (a) Find the numerical value of $E[XY]$.

   It helps to sketch the support of the pdf, which is the trianglar region of the unit square below the line $u = v$. By LOTUS,

$$E[XY] = \int_0^1 \int_0^u uv(4u^2) dv du = \int_0^1 2u^5 du = \frac{1}{3}.$$

   (b) The MMSE constant estimator is given by $\delta^* = E[Y]$. Using LOTUS again, yields

$$E[Y] = \int_0^1 \int_0^u v(4u^2) dv du = \int_0^1 2u^4 du = \frac{2}{5}.$$

(c) We begin by first finding the conditional pdf $f_{Y|X}(v|u)$. To this end, we first compute

$$f_X(u) = \begin{cases} \int_0^u 4u^2\,dv = 4u^3 & \text{if } 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

from which we get for $u \in (0,1]$,

$$f_{Y|X}(v|u) = \frac{f_{Y|X}(v|u)}{f_X(u)} = \begin{cases} \frac{1}{u} & \text{if } 0 \le v \le u \\ 0 & \text{else} \end{cases}$$

Therefore

$$g^*(u) = E[Y|X=u] = \int_0^u \frac{v}{u} dv = \frac{u}{2}$$

(ALTERNATIVELY, since for u fixed with $0 < u \le 1$, $f_{X,Y}(u,v)$ is constant in v over $v \in [0,u]$, it follows that the conditional distribution of Y is uniform over $[0,u]$, and therefore the conditional mean is $\frac{u}{2}$.)

11. (a) Let $c_{i,j} = \text{Cov}(X_i, X_j)$. Then
$\text{Cov}(X_1 + X_2, X_1 + X_3) = c_{1,1} + c_{1,3} + c_{2,1} + c_{2,3} = 5 + 0 + 2 + 2 = 9.$

(b) $\text{Cov}(X_2 - aX_1, X_1) = c_{2,1} - ac_{1,1} = 2 - 5a$, which is zero for $a = \frac{2}{5}$.

(c) $\text{Var}(X_i) = \text{Cov}(X_i, X_i) = 5$ for all i and $\text{Cov}(X_1, X_2) = 2$.
So $\rho_{X_1,X_2} = \frac{2}{\sqrt{5 \cdot 5}} = \frac{2}{5}$.

(d) $\text{Var}(X_1 + X_2 + X_3) = \text{Cov}(\sum_{i=1}^3 X_i, \sum_{j=1}^3 X_j) = 23$, because the covariance expands out to the sum of all entries in the covariance matrix.

12. (a) Note that $S = X_1 + \ldots + X_{25}$ where X_i denotes the weight of the i^{th} passenger. Then $E[S_{25}] = 25(150) = 3750$ and the standard deviation of S is $\sqrt{25(30)^2} = \sqrt{25}(30) = 150$.

(b) We apply the Gaussian approximation suggested by the central limit theorem to get: $P\{S \ge 4000\} = P\left\{\frac{S-3750}{150} \ge \frac{250}{150}\right\} \approx Q(\frac{250}{150}) = Q(1.666) = 1 - \Phi(1.666) = 1 - 0.9522 = 0.0478$.

13. (a) True, False, True

(b) True, False

(c) True,False

(d) False, False,True

Table 1: Φ function, the area under the standard normal pdf to the left of x.

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990