University of Illinois ECE 313: Conflict Final Exam Fall 2014

Tuesday, December 16, 2014, 7:00 p.m. — 10:00 p.m. Room 2013 ECE Building

Name (IN BLOCK LETTERS): ____

NetID: ______________________ **Signature:** ______________________________

Section: ☐ C, 10:00 a.m. ☐ D, 11:00 a.m. ☐ E, 1:00 p.m. ☐ B, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes are permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed. A table of the Φ function is at the end of the exam booklet.

The exam consists of thirteen problems worth a total of 200 points. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 12 points ____________
2. 12 points ____________
3. 12 points ____________
4. 12 points ____________
5. 12 points ____________
6. 12 points ____________
7. 18 points ____________
8. 20 points ____________
9. 10 points ____________
10. 18 points ____________
11. 20 points ____________
12. 12 points ____________
13. 30 points ____________

Total (200 points) ____________

1. **[12 points]** According to the Center for Disease Control (CDC), "Compared to nonsmokers, men who smoke are about 23 times more likely to develop lung cancer and women who smoke are about 13 times more likely."

   (a) (6 points) If you learn that a person is a woman who has been diagnosed with lung cancer, and you know nothing else about the person, what is the probability she is a smoker? In your solution, use the CDC information that roughly 15% of all women smoke.

   (b) (6 points) Suppose that in the USA, 15% of women are smokers, 18% of all adults are smokers, and half of adults are women. What fraction of adult smokers in the USA are women?

2. **[12 points]** (Suppose A, B, and C are three events such that

$$P(A) = P(B) = 0.3, \quad P(A^c|B) = \frac{2}{3}, \quad P(C) = 0.6, \quad P(ABC) = 0.1.$$

(a) (6 points) Fill in a Karnaugh map for A and B.

(b) (6 points) Fill in a Karnaugh map for A, B, and C satisfying the above assumptions such that $P(AC)$ is as small as possible. What is the minimum value of $P(AC)$?

3. **[12 points]** Let $(N_t, t \geq 0)$ and $(M_t, t \geq 0)$ be independent Poisson processes, both of rate λ. Let T be the first time *either* of the two processes is positive.

   (a) (6 points) Find the pdf of T.

   (b) (6 points) Find $E[T]$.

4. **[12 points]** A square is picked at random from a standard 8x8 chess board, with all possibilities having equal probability. (The side lengths of the square can be any of $1, \ldots, 8$.) Let X be the (random) length of the side of the picked square.

   (a) (6 points) Find the pmf of X.

   (b) (6 points) Find $E\left[\frac{1}{9-X}\right]$. Give the final numerical answer for full credit.

5. **[12 points]** Let $\mathcal{N}$ be a line communication network that consists of 2 links in series, each of which fails independently with probability p. Let $\mathcal{M}$ be a line communication network that consists of 4 links in series, each of which fails independently with probability $1/2$.

   (a) (6 points) Find the value of p for which the probability of failure of $\mathcal{N}$ is the same as the probability of failure of $\mathcal{M}$.

   (b) (6 points) Suppose you have 4 links that fail independently with probability $1/2$ each, and you can use them to build two communication networks. You have two design options: In the 3+1 design option, one network uses three links in series and the other just one link. In the 2+2 design option, both networks have two links in series. For which design is the probability that both networks fail smaller?

6. **[12 points]** Recall that if X is a random variable with mean μ and variance σ^2, then $Y = \frac{X-\mu}{\sigma}$ defines the standardized version of X. For each of the following two choices of distribution for X, carefully *sketch and label* the pmfs of X and Y. You should thus have a total of four sketches for this problem.

   (a) (6 points) X is the number generated by rolling a fair die. Carefully *sketch and label* the pmf of X *and* the pmf of Y. (Hint: The standard deviation of X is given by $\sigma \approx 1.7$.)

   (b) (6 points) X has the binomial distribution with parameters $n = 4$ and $p = 0.5$. Carefully *sketch and label* the pmf of X *and* the pmf of Y.

7. **[18 points]** Let $S = X + Y$, where X and Y are independent random variables, X is a Bernoulli random variable with parameter p with $0 < p < 1$, and $E[Y]$ is finite.

(a) (6 points) Express $\text{Cov}(X, S + Y)$ in terms of p.

(b) (6 points) If Y is a binomial random variable with parameters $n - 1$ and p, for some $n > 1$, what is the distribution of S?

(c) (6 points) Find $P\{S = 3\}$, provided that Y is a geometric random variable with parameter q with $0 < q < 1$.

8. **[20 points]** Let X and Y have joint pdf $f_{X,Y}(u,v) = \begin{cases} e^{-u-v} & u,v > 0 \\ 0 & \text{else} \end{cases}$.

(a) (7 points) Find the pdf of X^2.

(b) (7 points) Find $P\{X + Y \leq 1\}$.

(c) (6 points) Let $W = X^2$ and $Z = X + 2Y$. Find the joint pdf of W and Z.

9. **[10 points]** Let $(N_t, t \geq 0)$ be a Poisson process with parameter 2. Suppose it is known that $N_3 = 5$. Find the linear minimum mean square error (MMSE) estimator of N_1. As usual, show your work and/or justify your reasoning.

10. **[18 points]** Suppose random variables X and Y have the joint probability density function (pdf):

$$f_{XY}(u,v) = \begin{cases} 4u^2 & \text{if } 0 \le v \le u \le 1 \\ 0 & \text{else} \end{cases}$$

(a) Find the numerical value of $E[XY]$.

(b) (6 points) Find the numerical value of the minimum mean square error constant estimator, δ^*, of Y.

(c) (6 points) Find the function g^* so that $g^*(X)$ is the minimum mean square error estimator of Y based on observation X. Be as explicit as possible.

11. **[20 points]** Suppose the covariance matrix of a random vector $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ is $\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{pmatrix}$.

(Here, the ij^{th} entry of the matrix is $\text{Cov}(X_i, X_j)$. For example, $\text{Cov}(X_1, X_2) = 2$.)

(a) (5 points) Find $\text{Cov}(X_1 + X_2, X_1 + X_3)$.

(b) (5 points) Find a so that $X_2 - aX_1$ is uncorrelated with X_1.

(c) (5 points) Find the correlation coefficient, ρ_{X_1,X_2}

(d) (5 points) Find $\text{Var}(X_1 + X_2 + X_3)$.

12. **[12 points]** A particular airplane holds 25 passengers, and it generally flies with all seats full. Suppose the mean weight of a passenger is 150 pounds with a standard deviation of 30 pounds, and suppose the weights of the passengers are independent. Let S denote the total weight of all the passengers.

   (a) Find the mean and *standard deviation* of S. (Note that the standard deviation, in addition to the mean, is requested here. Carefully calculate the numerical values.)

   (b) Find the approximate numerical probability that the total weight of the passengers exceeds two tons (4,000 pounds). (Hint: Do you expect your answer to be less than or greater than 0.5? Is it?)

13. **[30 points]** (3 points per answer)

In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Consider events such that $P(ABC) = P(A)P(BC) > 0$ and $P(AC) = P(A)P(C)$.

TRUE	FALSE	
☐	☐	$P(B\|CA) = P(B\|C)$
☐	☐	$P(A\|BC) > P(A)$
☐	☐	If $P(B) < P(C)$, then $P(B\|C) < P(C\|B)$

(b) Suppose there are two hypotheses about the pmf for an observation X :

$$H_0: \quad p_0(1) = 0.6,\ p_0(2) = 0.3,\ p_0(3) = 0.1$$
$$H_1: \quad p_1(1) = 0.1,\ p_1(2) = 0.4,\ p_1(3) = 0.5.$$

Also suppose that the priors on the hypotheses are given by $\pi_0 = 2/3$ and $\pi_1 = 1/3$.

TRUE	FALSE	
☐	☐	The ML decision rule chooses H_0 if and only if $X = 1$.
☐	☐	The MAP decision rule always chooses H_0

(c) Suppose U and V are independent, with each being uniformly distributed over $[0, 1]$.

TRUE	FALSE	
☐	☐	$P\{U^2 + V^2 \leq 1\} = \frac{\pi}{4}$

TRUE	FALSE	
☐	☐	$P\{U^4 + V^4 \leq 1\} < \frac{\pi}{4}$

(d) Suppose X and Y are jointly Gaussian with mean zero and equal variances, such that the correlation coefficient ρ satisfies $\rho < 0$.

TRUE	FALSE	
☐	☐	$X + 2Y$ is independent of $X - 2Y$

TRUE	FALSE	
☐	☐	$P\{X + 2Y \geq 10\} > P\{X - 2Y \geq 10\}$

TRUE	FALSE	
☐	☐	$\widehat{E}[Y\|X]$ is a strictly decreasing function of X

Table 1: Φ function, the area under the standard normal pdf to the left of x.

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990