University of Illinois ECE 313: Final Exam Fall 2014

Monday, December 15, 2014, 7:00 p.m. — 10:00 p.m.

Sect. B, names A-O, 1013 ECE, names P-Z, 1015 ECE; Section C, names A-L, 1015 ECE; all others 112 Gregory

Name (IN BLOCK LETTERS): ______________________________

NetID: ________________ **Signature:** ________________________

Section: ☐ C, 10:00 a.m. ☐ D, 11:00 a.m. ☐ E, 1:00 p.m. ☐ B, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes are permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed. A table of the Φ function is at the end of the exam booklet.

The exam consists of twelve problems worth a total of 200 points. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 12 points ________
2. 12 points ________
3. 24 points ________
4. 12 points ________
5. 12 points ________
6. 18 points ________
7. 20 points ________
8. 10 points ________
9. 18 points ________
10. 20 oints ________
11. 12 points ________
12. 30 points ________

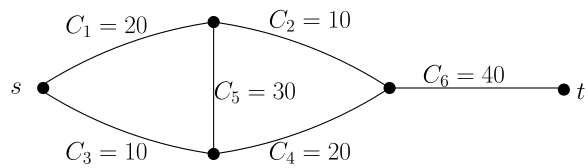
Total (200 points) ________

1. **[12 points]** A die is thrown repeatedly until the number three comes up twice (not necessarily consecutively). Let N be the (random) number of throws needed.

   (a) (6 points) Find $E[N]$.

   (b) (6 points) Find $E\left[\frac{1}{N-1}\right]$. Simplify your answer as much as possible.

2. **[12 points]** Consider the s-t network shown in the figure below, with the link capacities as labeled in the figure and the subscripts of C denoting the labels of the links.



Let F_i be the event that link i fails, $i = 1, 2, \ldots, 6$. The events $F_i, i = 1, 2, \ldots, 6$ are mutually independent and each event has probability p. Let Y denote the capacity of the network from s to t.

For each of the following parts, simplify your answers and write the expressions as polynomials in p and/or q, where $q = 1 - p$.

(a) (6 points) Find $P\{Y \geq 30\}$.

(b) (6 points) Find $P\{Y = 20\}$.

3. **[24 points]** A binary message source M_2 outputs *bytes* (8 bit words) such as 11010010 with every byte being equally likely. A quaternary message source M_4 produces words of length 8 with characters from the set $\{0, 1, 2, 3\}$, such as 32100313, with all such words being equally likely.

   (a) (6 points) What is the probability, p, that a word produced by M_4 is a byte, i.e., every character in the word belongs to the set $\{0, 1\}$?

   (b) (6 points) If a word is equally likely to come from each source, what is the probability it will be a byte?

   (c) (6 points) Given that the word is a byte, what is the probability it came from M_2?

   (d) (6 points) Machine M_4 produces a sequence of ten words, with each word being produced independently. Let X denote the number of bytes among the ten words. Describe the pmf of X, and find $E[X]$ and $\mathrm{Var}(X)$. You may express your answer in terms of p found in part (a).

4. **[12 points]** Recall that if X is a random variable with mean μ and variance σ^2, then $Y = \frac{X-\mu}{\sigma}$ defines the standardized version of X. For each of the following two choices of distribution for X, carefully *sketch and label* the pmfs of X and Y. You should thus have a total of four sketches for this problem.

   (a) (6 points) X is the number generated by rolling a fair die. Carefully *sketch and label* the pmf of X *and* the pmf of Y. (Hint: The standard deviation of X is given by $\sigma \approx 1.7$.)

   (b) (6 points) X has the binomial distribution with parameters $n = 4$ and $p = 0.5$. Carefully *sketch and label* the pmf of X *and* the pmf of Y.

5. **[12 points]** Let T have the exponential distribution with parameter λ. Express your answers to the following in terms of λ in the simplest form possible.

   (a) (6 points) Find $P(T \leq 12 | T \geq 5)$.

   (b) 6 points) Find $P\{x^2 + xT + 1 \geq 0 \text{ for all } x \in \mathbb{R}\}$.

6. **[18 points]** Let $S = X + Y$, where X and Y are independent random variables, X is a Bernoulli random variable with parameter p with $0 < p < 1$, and $E[Y]$ is finite.

   (a) (6 points) Express $\text{Cov}(X, S + Y)$ in terms of p.

   (b) (6 points) If Y is a binomial random variable with parameters $n - 1$ and p, for some $n > 1$, what is the distribution of S?

   (c) (6 points) Find $P\{S = 3\}$, provided that Y is a geometric random variable with parameter q with $0 < q < 1$.

7. **[20 points]** Let X and Y have joint pdf $f_{X,Y}(u,v) = \begin{cases} e^{-u-v} & u,v > 0 \\ 0 & \text{else} \end{cases}$.

(a) (7 points) Find the pdf of X^2.

(b) (7 points) Find $P\{X + Y \leq 1\}$.

(c) (6 points) Let $W = X^2$ and $Z = X + 2Y$. Find the joint pdf of W and Z.

8. **[10 points]** Let $(N_t, t \geq 0)$ be a Poisson process with parameter 2. Suppose it is known that $N_3 = 5$. Find the linear minimum mean square error (MMSE) estimator of N_1. As usual, show your work and/or justify your reasoning.

9. **[18 points]** Suppose X and Y are random variables with joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{u^2v^2} & u \geq 1, v \geq 1 \\ 0 & \text{else} \end{cases}.$$

(a) (6 points) Find the conditional pdf of Y given $X = 5$, $f_{Y|X}(v|5)$, for all $v \in \mathbb{R}$.

(b) (6 points) Find $E\left[\frac{1}{X}\right]$.

(c) (6 points) Find $P\{Y \geq 5X\}$. Clearly show your work and simplify your final answer.

10. **[20 points]** Suppose the covariance matrix of a random vector $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ is $\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{pmatrix}$.

(Here, the ij^{th} entry of the matrix is $\mathrm{Cov}(X_i, X_j)$. For example, $\mathrm{Cov}(X_1, X_2) = 2$.)

(a) (5 points) Find $\mathrm{Cov}(X_1 + X_2, X_1 + X_3)$.

(b) (5 points) Find a so that $X_2 - aX_1$ is uncorrelated with X_1.

(c) (5 points) Find the correlation coefficient, ρ_{X_1,X_2}

(d) (5 points) Find $\mathrm{Var}(X_1 + X_2 + X_3)$.

11. **[12 points]** A particular airplane holds 25 passengers, and it generally flies with all seats full. Suppose the mean weight of a passenger is 150 pounds with a standard deviation of 30 pounds, and suppose the weights of the passengers are independent. Let S denote the total weight of all the passengers.

    (a) Find the mean and *standard deviation* of S. (Note that the standard deviation, in addition to the mean, is requested here. Carefully calculate the numerical values.)

    (b) Find the approximate numerical probability that the total weight of the passengers exceeds two tons (4,000 pounds). (Hint: Do you expect your answer to be less than or greater than 0.5? Is it?)

12. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose that X and Y are independent standard Gaussian random variables.

TRUE	FALSE	
☐	☐	$X + Y$ is a standard Gaussian random variable.
☐	☐	$P(X \leq 3 \| Y \geq 0) = P(Y \leq 3 \| X \geq 0)$.
☐	☐	Let $Z = X^2$. The variable Z is standard Gaussian.

(b) Let $X_1, \ldots, X_m$ be independent random variables, each with the binomial distribution with parameters 10 and p, where $0 < p < 1$, and let $S_m = X_1 + \ldots + X_m$.

TRUE	FALSE	
☐	☐	S_m has a binomial distribution
☐	☐	$\lim_{m\to\infty} P\left\{\frac{S_m}{m} \geq 10p(1-p)\right\} = 1$

(c)

TRUE	FALSE	
☐	☐	If N is a geometric random variable with parameter p, then $P(N > 10 \| N > 2) = P(N > 8)$.
☐	☐	Suppose X is binomial (n, p), Y is binomial (m, p), and X and Y are independent. Then $X + Y$ is binomial $(n + m, 2p)$.

(d) Suppose U and V are jointly Gaussian with equal variances, mean zero, and correlation coefficient ρ with $\rho > 0$.

TRUE	FALSE	
☐	☐	$U + V$ is independent of $U - V$.
☐	☐	$P\{U + V \geq 10\} > P\{U \geq 10\}$.
☐	☐	$P\{U + V \geq 10\} > P\{U - V \geq 10\}$.

Table 1: Φ function, the area under the standard normal pdf to the left of x.

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990