University of Illinois Fall 2014

ECE 313: Hour Exam I

Wednesday, October 8, 2014
7:00 p.m. — 8:15 p.m.
Section B in 151 Everitt Lab, Sections C & E in 141 Loomis Lab, Section D in 100 MSEB

1. (a) There are $\binom{n}{2}$ possibilities for which two cells fail, and $n-1$ possibilities for two neighboring cells to fail: 1 and 2, 2 and 3, ..., $n-1$ and n. So the probability is

$$\frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}$$

(b) Let A be the event at least one of the two failures is among the first four cells and B be the event both failures are among the first four cells. Then $|B| = \binom{4}{2} = 6$, because there are $\binom{4}{2}$ ways to select two of the first four cells to fail, and $|A| = \binom{4}{2} + 4(n-4) = 4n-10$ because there are $\binom{4}{2}$ ways two select two positions out of the first four, and $4(n-4)$ ways to select a pair of positions with one of those being among the first four and the other among the other n-4 positions. Thus,

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{|AB|}{|A|} = \frac{|B|}{|A|} = \frac{6}{4n-10}.$$

2. (a) If p denotes the parameter of the geometric distribution X has, then $E[X] = \frac{1}{p} = 1+\theta$. Therefore, $p = \frac{1}{1+\theta}$, or, equivalently, $\theta = \frac{1}{p} - 1$. Moreover, $p_X(10) = (1-p)^9 p$. So $\hat{\theta}_{\rm ML} = \frac{1}{p^*} - 1$, where p^* maximizes $p_X(10)$. To find p^*, we examine the derivative of either $p_X(10)$ or $\ln p_X(10)$ with respect to p (see Example 2.8.3 in the notes). The result is $p^* = \frac{1}{10} = 0.1$, so that $\hat{\theta}_{\rm ML} = 9$.

(b) The memorylessness property of the geometric distribution yields
$P(X = 12|X > 10) = P\{X = 2\} = (1-0.2)(0.2) = 0.16$.

3. (a) The possible values of X are 0,1, or 2.
$p_X(0) = \frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$,
$p_X(2) = \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{5}{15}$, and
$p_X(1) = \frac{4}{10} \cdot \frac{6}{9} + \frac{6}{10} \cdot \frac{4}{9} = \frac{48}{90} = \frac{8}{15}$. (Or could use the fact the three probabilities add to one. Or could observe there are $4 \times 6 = 24$ ways to select one orange and one blue t-shirt from the first bag out of $\binom{10}{2} = 45$ ways to select two t-shirts from the first bag.)

(b) Let B denote the event the t-shirt chosen from the second bag is blue. Then, by the law of total probability,

$$\begin{aligned} P(B) &= P(B|X=2)P(X=2) + P(B|X=1)P(X=1) + P(B|X=0)P(X=0) \\ &= \frac{2}{3} \cdot \frac{5}{15} + \frac{1}{3} \cdot \frac{8}{15} + \frac{0}{3} \cdot \frac{2}{15} = \frac{18}{45}. \end{aligned} \quad (1)$$

(c) Using the definition of conditional probability, $P(X = 2|B)$ is the first term in (1) divided by the whole sum in (1):

$$P(X = 2|B) = \frac{2 \cdot 5}{2 \cdot 5 + 1 \cdot 8 + 0 \cdot 2} = \frac{10}{18} = \frac{5}{9}.$$

Note: The combination of using the law of total probability in part (b) and the definition of conditional probability in part (c) amounts to Bayes formula.

4. (a) Using the method based on the Chebychev inequality, the width of the confidence interval is $\frac{a}{\sqrt{n}}$,with $n = 200,000$ and

$$1 - \frac{1}{a^2} = 0.95 \quad \Longrightarrow \quad a = \sqrt{20}.$$

Thus the width is $\frac{\sqrt{20}}{\sqrt{200{,}000}} = \frac{1}{100} = 0.01$.

(b) To shrink $\frac{a}{\sqrt{n}}$ by a factor of 10 requires increasing n by a factor of 10^2. So the number of trials needed equals $200,000 \times 10^2 = 2 \times 10^7$.

5. (a) The easiest way to solve the problem is to use the table of conditional probabilities (aka likelihood matrix):

Table 1: The likelihood matrix and ML decision rule

	$P\{X = 1\|H\}$	$P\{X = 2\|H\}$	$P\{X = 3\|H\}$	$P\{X = 4\|H\}$
H_0	1/3	1/3	1/3	0
H_1	1/2	1/8	1/8	1/4
ML	H_1	H_0	H_0	H_1

(b) From the table, $p_{\text{false alarm}} = P(\text{say } H_1|H_0 \text{ true}) = P(X = 1|H_0) = 1/3$.

(c) Multiplying the row for H_i in the likelihood matrix by π_i for both values of i, we obtain:

Table 2: The joint probability matrix and MAP decision rule

	$P\{X = 1, H\}$	$P\{X = 2, H\}$	$P\{X = 3, H\}$	$P\{X = 4, H\}$
H_0	2/9	2/9	2/9	0
H_1	1/6	1/24	1/24	1/12
MAP	H_0	H_0	H_0	H_1

(d) Since $P\{X = 4|H_0\} = 0$, no value of $\pi_1 > 0$ can lead to a decision rule that always favors H_0.

6. (a) Let A_i denote the event you like book i. By the union bound, the probability you will like at least one of the books is bounded above: $P(A_1 \cup A_2) \leq P(A_1) + P(A_2) = 0.9$. Thus, the probability of the complement of this event, i.e. you will like neither of the books, is at least 0.1.

(b) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2) = 0.5 + 0.4 - 0.3 = 0.6$. By using de Morgan's law, you consequently have

$$P(A_1^c A_2^c) = P((A_1 \cup A_2)^c) = 1 - P(A_1 \cup A_2) = 0.4.$$