University of Illinois **Fall 2013**

ECE 313: Problem Set 12
Functions of Random Variables, Covariance and Correlation

Due:	Wednesday, December 4 at 6 p.m.
Reading:	*313 Course Notes* Sections 4.1-4.9
Reminder:	The Final Exam is on Monday, December 16, 8:00 a.m. – 11:00 a.m. **Location:** Sections D (meets 11 am) and E (meets 1 pm) 100 MSEB Sections X (meets 9am) and C (meets 10am) 151 EL **Two** two-sided 8.5"×11" sheet of notes are allowed, with font size no smaller than 10 pt or equivalent handwriting. Bring your student ID.

1. **[Circularly symmetric joint pdf]**
   Problem 4.15 of the *Course Notes*

2. **[Randomly routing to two different servers]**
   A web site has two servers. When a request comes in, the router mentally flips a coin and routes the request to Server I if the result is Heads and to Server II if the result is Tails. Each coin flip is independent of all others and results in Head with probability p, $0 < p < 1$. Let $\mathbb{N}$ denote the number of requests that arrive in one minute. We assume that $\mathbb{N}$ is a Poisson random variable with parameter λ. Let X the number of requests routed to Server I during this minute. Thus, *conditioned* on $\mathbb{N} = n$, $n > 0$, $\mathbb{X}$ is a *binomial* random variable with parameters (n, p). Let $\mathbb{Y}$ denote the number of requests routed to Server II. Then, given that $\mathbb{N} = n$, Y is a binomial random variable with parameters $(n, 1-p)$. In fact, given that $\mathbb{N} = n$, $\mathbb{Y} = n - \mathbb{X}$ and $\mathbb{X}$ are very much (conditionally) *dependent* random variables.

   (a) For $0 \le k \le n$, the *conditional* pmf of $\mathbb{X}$ given $\mathbb{N} = n$ is $p_{\mathbb{X}|N}(k \mid \mathbb{N} = n) = \binom{n}{k} p^k (1-p)^{n-k}$. What is the *joint* pmf $p_{\mathbb{X},\mathbb{N}}(k, n)$? Be sure to specify the value of $p_{\mathbb{X},\mathbb{N}}(k, n)$ for *all* $k, n \ge 0$, and explain why for any given k, $p_{\mathbb{X},\mathbb{N}}(k, n)$ is nonzero only for $n \ge k$.

   (b) What is the *marginal* pmf $p_{\mathbb{X}}(k)$? Be sure to specify the value of the marginal pmf for all $k \ge 0$.

   (c) Given that Server I received $\mathbb{X} = k$ requests, what is the *conditional* pmf $p_{\mathbb{N}|\mathbb{X}}(n \mid \mathbb{X} = k)$ of $\mathbb{N}$?

   (d) We have noted that $\mathbb{X}$ and $\mathbb{Y}$ are *conditionally dependent* random variables given that $\mathbb{N} = n$. Prove that $\mathbb{X}$ and $\mathbb{Y}$ are *unconditionally independent* random variables. That is, show that for all $i, j \ge 0$, $p_{\mathbb{X},\mathbb{Y}}(i, j) = p_{\mathbb{X}}(i) p_{\mathbb{Y}}(j)$.

3. **[Drill on covariance and correlation]**
   Let $\mathsf{E}[\mathbb{X}] = 1, \mathsf{E}[\mathbb{Y}] = 4, \mathsf{var}(\mathbb{X}) = 4, \mathsf{var}(\mathbb{Y}) = 9$, and $\rho_{\mathbb{X},\mathbb{Y}} = 0.1$.

   (a) If $\mathbb{Z} = 2(\mathbb{X} + \mathbb{Y})(\mathbb{X} - \mathbb{Y})$, what is $\mathsf{E}[\mathbb{Z}]$?

   (b) If $\mathbb{T} = 2\mathbb{X} + \mathbb{Y}$ and $\mathbb{U} = 2\mathbb{X} - \mathbb{Y}$, what is $\mathsf{cov}(\mathbb{T}, \mathbb{U})$?

   (c) Find the mean and variance of $\mathbb{W} = 3\mathbb{X} + \mathbb{Y} + 2$.

4. **[More drill on covariance and correlation]**
   This problem has three independent parts. Do not apply the numbers from one part to the others.

   (a) If $\mathsf{var}(\mathbb{X} + \mathbb{Y}) = 36$ and $\mathsf{var}(\mathbb{X} - \mathbb{Y}) = 64$, what is $\mathsf{cov}(\mathbb{X}, \mathbb{Y})$? If you are also told that $\mathsf{var}(\mathbb{X}) = 3 \cdot \mathsf{var}(\mathbb{Y})$, what is $\rho_{\mathbb{X},\mathbb{Y}}$?

   (b) If $\mathsf{var}(\mathbb{X} + \mathbb{Y}) = \mathsf{var}(\mathbb{X} - \mathbb{Y})$, are $\mathbb{X}$ and $\mathbb{Y}$ uncorrelated ?

   (c) If $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$, are $\mathbb{X}$ and $\mathbb{Y}$ uncorrelated ?

5. **[Conditional densities and means]**
The joint pdf of random variables $\mathbb{X}$ and $\mathbb{Y}$ is given

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} \dfrac{e^{-u/v}e^{-v}}{v}, & 0 < u < \infty, 0 < v < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are $\mathbb{X}$ and $\mathbb{Y}$ independent random variables?

(b) Find $f_{\mathbb{X}|\mathbb{Y}}(u \mid \mathbb{Y} = v)$, the conditional density of $\mathbb{X}$ given that $\mathbb{Y} = v$.

(c) What is the conditional mean $\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = v]$ of $\mathbb{X}$ given that $\mathbb{Y} = v$?

(d) Since the value of the conditional mean of $\mathbb{X}$ given the value of $\mathbb{Y}$ depends on the value of $\mathbb{Y}$, that is, it is a *function* of $\mathbb{Y}$, the conditional mean of $\mathbb{X}$ can also be regarded as a random variable; in fact a function of the random variable $\mathbb{Y}$. It is conventional to denote this random variable by $\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]$ with the meaning that if $\mathbb{Y}$ happens to have value α, then this *function* of $\mathbb{Y}$ has value $\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = \alpha]$. Use your answer of part (c) to express $\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]$ as a function of $\mathbb{Y}$ without using the E operator.

(e) The *law of iterated expectation* (with the unfortunate acronym LIE) claims that the expected value of $\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]$ equals $\mathsf{E}[\mathbb{X}]$, that is,

$$\mathsf{E}\Big[\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]\Big] = \mathsf{E}[\mathbb{X}].$$

Note very carefully: the right side is $\mathsf{E}[\mathbb{X}]$; **NOT** $\mathsf{E}[\mathbb{Y}]$ even though $\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]$ is a function of $\mathbb{Y}$, not of $\mathbb{X}$.. In this particular case, of course, since $\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]$ happens to *equal* $\mathbb{Y}$, the LIE tells us that $\mathsf{E}\Big[\mathsf{E}[\mathbb{X} \mid \mathbb{Y}]\Big] = \mathsf{E}[\mathbb{X}]$ has the same value as $\mathsf{E}[\mathbb{Y}]$.

So, what *is* $\mathsf{E}[\mathbb{Y}]$? And what is the value of $\mathsf{E}[\mathbb{X}]$?

(f) Calculate $\mathsf{E}[\mathbb{X}]$ directly from the joint pdf of $\mathbb{X}$ and $\mathbb{Y}$. Do you get the same answer as in part (f)?

(g) There is no part (g).