University of Illinois **Fall 2013**

ECE 313: Problem Set 8
Gaussian Random Variables, ML Parameter Estimation, Function of a Random Variable

Due: Wednesday, October 30 at 6 p.m.
Reading: *313 Course Notes* Sections 3.6-3.8

1. [**A puzzle about the Gaussian distribution**]
Suppose that $\mathbb{X} \sim \mathsf{N}(\mu.\sigma^2)$ and that $P\{X > 20.6\} = P\{X \leq -18.6\} = 0.025$. What are the values of the mean μ and standard deviation σ?

2. [**Working with a table of the unit Gaussian distribution function**]
The width of a metal trace on a circuit board is modelled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

   (a) Traces that fail to meet the requirement that the width be in the range 0.9 ± 0.005 microns are said to be defective. What percentage of traces are defective?

   (b) A new manufacturing process that produces smaller variations in trace widths is to be designed so as to have no more than 1 defective trace in 100. What is the maximum value of σ for the new process if the new process achieves the goal?

3. [**DeMoivre-Laplace approximation to central term of binomial distribution**]
Let n be a positive *even* integer, and let $\mathbb{X}$ be a binomial random variable with parameters $(n, 0.5)$. This problem focuses on $P\{\mathbb{X} = \frac{n}{2}\}$. The continuity correction for approximating the binomial distribution by the normal distribution begins by writing this same probability as $P\{\frac{n-1}{2} \leq \mathbb{X} \leq \frac{n+1}{2}\}$.

   (a) Using the continuity correction, find the normal approximation to $P\{\mathbb{X} = \frac{n}{2}\}$. Your answer should involve n and the standard normal CDF $\Phi(x)$.

   (b) Find the constant c such that $\sqrt{n}P\{\mathbb{X} = \frac{n}{2}\} \to c$ as $n \to \infty$, assuming you can replace $P\{\mathbb{X} = \frac{n}{2}\}$ by its normal approximation found in part (a). This suggests that $P\{\mathbb{X} = \frac{n}{2}\} \approx \frac{c}{\sqrt{n}}$ for large n. (Hint: Since $\Phi(x)$ is differentiable for all x, then $\frac{\Phi(h)-\Phi(0)}{h} \to \frac{d}{dx}\Phi(x)\big|_{x=0} = \Phi'(0)$ as $h \to 0$.)

   (c) For $n = 30$, compute the exact value of $P\{\mathbb{X} = \frac{n}{2}\}$, the approximation found in part (a), and the approximation found in part (b).

4. [**ML estimation of a parameter of a uniform density**]
Problem 3.22, p. 151, of the *Course Notes*

5. [**Current through a semiconductor diode**]
The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathbb{V}$ with pdf

$$f_{\mathbb{V}}(u) = 0.5\exp(-|u|), \quad -\infty < u < \infty.$$

So, the current $\mathbb{I} = I_0(\exp(\mathbb{V}) - 1)$ is also a continuous random variable.

   (a) What values can $\mathbb{I}$ take on?

   (b) Find the CDF of $\mathbb{I}$.

   (c) Find the pdf of $\mathbb{I}$.

6. **[An A/D converter]**
This is a variation of Problem 3.26 of the *Course Notes*.

A signal $\mathbb{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathbb{Y}$ (where $\mathbb{Y} = \alpha$ if $\mathbb{X} > 0$ and $\mathbb{Y} = -\alpha$ if $\mathbb{X} \leq 0$) is used. Note that $\mathbb{Y}$ is a *discrete* random variable.

(a) What is the pmf of $\mathbb{Y}$?

(b) The *squared error* in representing $\mathbb{X}$ by $\mathbb{Y}$ is $\mathbb{Z} = \begin{cases} (\mathbb{X} - \alpha)^2, & \text{if } \mathbb{X} > 0, \\ (\mathbb{X} + \alpha)^2, & \text{if } \mathbb{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathbb{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathbb{Z}]$. Use LOTUS to easily calculate $\mathsf{E}[\mathbb{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathbb{Z}]$.
Hint: Before you start evaluating the integrals that LOTUS gives you for $\mathsf{E}[\mathbb{Z}]$, write down the *integral* that you would use to compute the variance of X. Also, compute $\frac{d}{du}e^{-u^2/2}$, and have these things in front of you. It will make finding $\mathsf{E}[\mathbb{Z}]$ EZ, or EZier.

(c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathbb{X}$ to the nearest integer $\mathbb{W}$ in the range -3 to $+3$. Thus, $\mathbb{W} = 3$ if $\mathbb{X} \geq 2.5$, $\mathbb{W} = 2$ if $1.5 \leq \mathbb{X} < 2.5$, $\mathbb{W} = 1$ if $0.5 \leq \mathbb{X} < 1.5$, $\cdots$, $\mathbb{W} = -3$ if $\mathbb{X} < -2.5$. Note that $\mathbb{W}$ is also a discrete random variable. Find the pmf of $\mathbb{W}$.

(d) The output of the A/D converter is a 3-bit 2's complement representation of $\mathbb{W}$. Suppose that the output is $(\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_0)$. What is the pmf of $\mathbb{Z}_2$? the pmf of $\mathbb{Z}_1$? the pmf of $\mathbb{Z}_0$? Note that $(1, 0, 0)$ which represents -4 is not one of the possible outputs from this A/D converter.