University of Illinois Fall 2013

ECE 313: Second MidTerm Exam

Monday November 18, 2013
7:00 p.m. — 8:20 p.m.

1. **[6 points]** The random variable $\mathbb{X}$ is uniformly distributed over the interval $[-1, 1]$. What is the value of $\mathsf{var}(\mathbb{X}^3)$?

   Solution:
   $$\mathsf{var}(\mathbb{X}^3) = \mathsf{E}[\mathbb{X}^6] - \left(\mathsf{E}[\mathbb{X}^3]\right)^2 = \mathsf{E}[\mathbb{X}^6] = \int_{-1}^{1} x^6 \cdot \frac{1}{2} dx = \frac{1}{7}.$$

2. **[6 points]** $\mathbb{X}$ is a Gaussian random variable (mean 60, variance 400) that models the average daily temperature (in °F) in a certain city.
   What is the probability that the average daily temperature is below 0°F?

   Solution: $P\{\mathbb{X} < 0\} = \Phi\left(\frac{0-60}{20}\right) = \Phi(-3) = 1 - \Phi(3) = 1 - 0.9987 = 0.0013$

3. **[6 points]** The joint pdf of random variables $\mathbb{X}$ and $\mathbb{Y}$ is
   $$f_{\mathbb{X},\mathbb{Y}}(u, v) = \begin{cases} a, & 0 \le u \le 1, 0 \le v \le 1, u + v \le 1, \\ b, & 0 \le u \le 1, 0 \le v \le 1, 1 < u + v \le 2. \end{cases}$$

   What is the numerical value of $a + b$?

   Solution: The joint pdf has constant value a on the triangle of area $\frac{1}{2}$ with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ and value b on the rest of the unit square. Thus, the integral of the pdf over the unit square is $a \times \frac{1}{2} + b \times \frac{1}{2}$ and since this must equal 1, we have that $a + b = 2$.

4. **[24 points]** $\mathbb{X}$ denotes a continuous random variable with pdf $f_{\mathbb{X}}(u)$ satisfying
   $$f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u) \text{ for all } u, \quad -\infty < u < \infty.$$

   Suppose that $\mathsf{var}(\mathbb{X}) = 9$. Let $\mathbb{Y} = |\mathbb{X}|$ and $\mathbb{Z} = -\mathbb{X}$, and consider the eight statements below for all random variables satisfying these conditions.

   Mark each statement as TRUE or FALSE.
   You do not need to provide any justification for your answers.

   Each correct answer will earn you 3 points. In order to discourage guessing, 3 points will be **deducted** from your score for each incorrect answer. Statements with both boxes unmarked will not affect your score. If you mark one box and then change your mind and mark the other, please be sure to indicate clearly what your final answer is.

   A net negative score on this problem will reduce your total exam score.

TRUE FALSE

☐ ☐ $P\{\mathbb{X} > \alpha\} = F_{\mathbb{X}}(-\alpha)$ for all $\alpha, \ -\infty < \alpha < \infty$.

☐ ☐ $F_{\mathbb{Y}}(v) = 2F_{\mathbb{X}}(v) - 1$ for $v \geq 0$, and 0 for $v < 0$.

☐ ☐ $F_{\mathbb{Z}}(w) = F_{\mathbb{X}}(-w)$ for all $w, \ -\infty < w < \infty$.

☐ ☐ $f_{\mathbb{Z}}(w) = f_{\mathbb{X}}(-w)$ for all $w, \ -\infty < w < \infty$.

☐ ☐ $\mathsf{E}[\mathbb{Y}^2] = 9$.

☐ ☐ $\mathsf{E}[\mathbb{Y}] = 3$.

☐ ☐ $\mathsf{var}(\mathbb{Y}) < 9$.

☐ ☐ $\mathbb{X}$ and $\mathbb{Y}$ are independent random variables.

Solution:
TRUE by symmetry of pdf: $P\{\mathbb{X} > \alpha\} = P\{\mathbb{X} < -\alpha\} = F_{\mathbb{X}}(-\alpha)$
TRUE by symmetry of pdf: $P\{|\mathbb{X}| < v\} = P\{-v < \mathbb{X} < v\} = F_{\mathbb{X}}(v) - F_{\mathbb{X}}(-v)$
FALSE: $F_{\mathbb{X}}(-w)$ is a decreasing function of w and cannot be a valid CDF.
TRUE: $F_{\mathbb{Z}}(w) = 1 - F_{\mathbb{X}}(-w)$ has derivative $f_{\mathbb{X}}(-w)$.
TRUE: $\mathsf{E}[\mathbb{Y}^2] = \mathsf{E}[\mathbb{X}^2] = \mathsf{var}(\mathbb{X}) + (\mathsf{E}[\mathbb{X}])^2 = 9$ since $\mathsf{E}[\mathbb{X}] = 0$.
FALSE: It would imply that $\mathsf{var}(\mathbb{Y}) = 0$ which is not true.
TRUE: $\mathsf{var}(\mathbb{Y}) = \mathsf{E}[\mathbb{Y}^2] - (\mathsf{E}[\mathbb{Y}])^2 = 9 - (\mathsf{E}[\mathbb{Y}])^2 < 9$.
FALSE: knowing the value of $\mathbb{X}$ tells us the exact value of $\mathbb{Y}$,

5. **[18 points]** Let $\mathbb{X}$ denote the lifetime (measured in years) of a mobile phone drawn at random from a set of mobile phones that *appear* to be identical, but are not. For *half* the phones in the set, $\mathbb{X}$ is a *discrete* random variable taking on value 1 with probability $\frac{1}{3}$ and value 3 with probability $\frac{2}{3}$. The lifetimes of the other phones are *exponentially* distributed random variables with parameter λ.

   A phone is chosen at random from this set of mobile phones.

   (a) **[8 points]** Given that the chosen phone is working at the end of 4 years, what is the conditional probability that it will work for at least 3 more years?
   Solution: Since we are given that $\mathbb{X} > 3$, the lifetime of the phone must be exponentially distributed with parameter λ. From the memoryless property of exponential random variables we have that

$$P\{\mathbb{X} > 7 \mid \mathbb{X} > 4\} = \frac{P\{\mathbb{X} > 7\}}{P\{\mathbb{X} > 4\}} = \frac{e^{-7\lambda}}{e^{-4\lambda}} = e^{-3\lambda}.$$

More formally, let A denote the event that the phone's lifetime is the discrete random variable. Then, for *any* $T \geq 3$,

$$P\{\mathbb{X} > T\} = P\{\mathbb{X} > T \mid A\}P(A) + P\{\mathbb{X} > T \mid A^c\}P(A^c) = 0 \times \frac{1}{2} + e^{-\lambda T} \times \frac{1}{2} = \frac{1}{2}e^{-\lambda T}$$

and so $P\{\mathbb{X} > 7 \mid \mathbb{X} > 4\} = \dfrac{P\{\mathbb{X} > 7\}}{P\{\mathbb{X} > 4\}} = \dfrac{\frac{1}{2}e^{-7\lambda}}{\frac{1}{2}e^{-4\lambda}} = e^{-3\lambda}$ as before.

(b) **[10 points]** Given that the phone is working at the end of two years, what is the conditional probability that it will work for *at least three* more years?

Solution: As in part (a), A denotes the event that the phone's lifetime is the discrete random variable. Then,

$$\begin{aligned} P\{\mathbb{X} > 2\} &= P\{\mathbb{X} > 2 \mid A\}P(A) + P\{\mathbb{X} > 2 \mid A^c\}P(A^c) \\ &= P\{\mathbb{X} = 3 \mid A\}P(A) + P\{\mathbb{X} > 2 \mid A^c\}P(A^c) \\ &= \frac{2}{3} \times \frac{1}{2} + e^{-2\lambda} \times \frac{1}{2} \\ &= \frac{1}{3} + \frac{1}{2}e^{-2\lambda}. \end{aligned}$$

Also, from the result $P\{\mathbb{X} > T\} = \frac{1}{2}e^{-\lambda T}$ for $T \geq 3$ obtained in part (a), we have that $P\{\mathbb{X} > 5\} = \frac{1}{2}e^{-5\lambda}$. Hence, $P\{\mathbb{X} > 5 \mid \mathbb{X} > 2\} = \dfrac{P\{\mathbb{X} > 5\}}{P\{\mathbb{X} > 2\}} = \dfrac{\frac{1}{2}e^{-5\lambda}}{\frac{1}{3} + \frac{1}{2}e^{-2\lambda}} = \dfrac{e^{-5\lambda}}{\frac{2}{3} + e^{-2\lambda}}$

6. **[22 points]** Let $\mathbb{X}$ denote a continuous random variable. When hypothesis H_0 is true, $\mathbb{X}$ is uniformly distributed on $[50, 100]$. When H_1 is true, $\mathbb{X}$ is uniformly distributed on $[0, 60]$. Recall that for a decision rule D, the *decision region* $\Gamma_{i,D}$ is the set of real numbers such that the decision is in favor of H_i whenever the observed value of $\mathbb{X}$ belongs to $\Gamma_{i,D}$.

(a) **[5 points]** Find the decision region $\Gamma_{0,\text{ML}}$ for the *maximum-likelihood* (ML) decision rule.

Solution: $\Gamma_{0,\text{ML}}$ is the set of all real numbers for which $f_0(u) > f_1(u)$. Since $f_0(u)$ has value $\frac{1}{50}$ for $u \in [50, 100]$ while $f_1(u)$ has value $\frac{1}{60} < \frac{1}{50}$ for $u \in [0, 60]$, we get that $\Gamma_{0,\text{ML}} = [50, 100]$.

(b) **[5 points]** What is the value of $P_{\text{miss,ML}}$, the probability of *missed detection* (also known as theType II error probability), for the maximum-likelihood decision rule?

Solution: A missed detection occurs in the ML decision rule if $\mathbb{X} \in \Gamma_{0,\text{ML}}$ when H_1 is the true hypotheses. Thus,

$$P_{\text{miss,ML}} = \int_{\Gamma_{0,\text{ML}}} f_1(u)\,du = \int_{50}^{60} \frac{1}{60}\,du = \frac{1}{6}.$$

Note incidentally that $P_{\text{falsealarm,ML}} = \displaystyle\int_{\Gamma_{1,\text{ML}}} f_0(u)\,du = \int_0^{50} 0\,du = 0.$

(c) **[6 points]** If the *prior probabilities* π_0 and π_1 of the hypotheses are such that $\pi_1 = 2\pi_0$, what is $\Gamma_{1,\text{MAP}}$ for the *maximum a posteriori* (MAP) decision rule? Remember that the MAP decision rule is also the *minimum-error-probability* (MEP) decision rule.

Solution: We have $\pi_1 = \frac{2}{3}$ and $\pi_0 = \frac{1}{3}$. $\Gamma_{1,\text{MAP}}$ is the set of real numbers u such that $\pi_1 f_1(u) > \pi_0 f_0(u)$. Since $\pi_1 f_1(u)$ has value $\frac{2}{3} \times \frac{1}{60} = \frac{1}{90}$ for $u \in [0, 60]$ while $\pi_0 f_0(u)$ has value $\frac{1}{3} \times \frac{1}{50} = \frac{1}{150}$ for $u \in [50, 100]$, it follows that $\Gamma_{1,\text{MAP}} = [0, 60]$.

(d) **[6 points]** What is the (average) error probability $P_{e,\text{MAP}}$ of the MAP decision rule?

Solution: Since $\Gamma_{0,\text{MAP}} = (60, 100]$, $P_{\text{miss,MAP}} = \int_{\Gamma_{0,\text{MAP}}} f_1(u)\,du = \int_{60}^{100} 0\,du = 0$.

On the other hand, $P_{\text{falsealarm,MAP}} = \int_{\Gamma_{1,\text{MAP}}} f_0(u)\,du = \int_{50}^{60} \frac{1}{50}\,du = \frac{1}{5}$. Consequently,

$$P_{e,\text{MAP}} = \pi_0 P_{\text{falsealarm,MAP}} + \pi_1 P_{\text{miss,MAP}} = \frac{1}{3} \times \frac{1}{5} + \frac{2}{3} \times 0 = \frac{1}{15}.$$

In contrast,

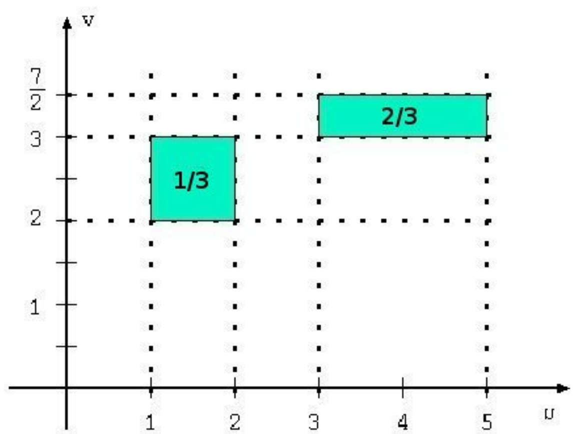
$$P_{e,\text{ML}} = \pi_0 P_{\text{falsealarm,ML}} + \pi_1 P_{\text{miss,ML}} = \frac{1}{3} \times 0 + \frac{2}{3} \times \frac{1}{6} = \frac{1}{9} > \frac{1}{15} = P_{e,\text{MAP}}$$

which of course is to be expected.

7. **[18 points]** Let $\mathbb{X}$ and $\mathbb{Y}$ be jointly continuous random variables with joint pdf

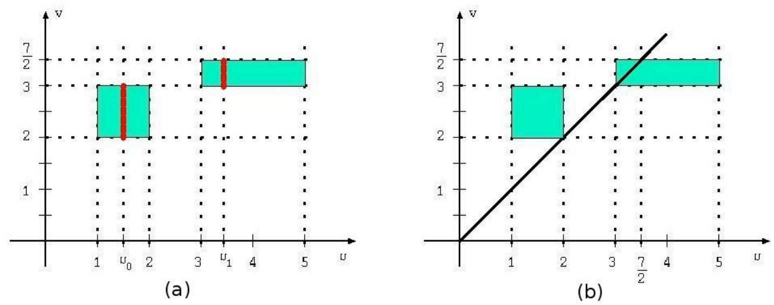
$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} \frac{1}{3} & 1 \le u \le 2,\ 2 \le v \le 3 \\ \frac{2}{3} & 3 \le u \le 5,\ 3 \le v \le \frac{7}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(a) **[6 points]** Are $\mathbb{X}$ and $\mathbb{Y}$ independent?
Solution: The support of the joint pdf is as shown below.



The supports of $f_{\mathbb{X}}(u)$ and $f_{\mathbb{Y}}(v)$ are $[1,2] \cup [3,5]$ and $[2, 3.5]$ respectively whose product is a set of 2 rectangles *larger* than the 2 are the support of $f_{\mathbb{X},\mathbb{Y}}(u,v)$, Hence, $\mathbb{X}$ and $\mathbb{Y}$ are *dependent* random variables.

(b) **[6 points]** Find the marginal pdf $f_{\mathbb{X}}(u)$.
To obtain full credit, you must specify the value of $f_{\mathbb{X}}(u)$ for all $u, -\infty < u < \infty$.
Solution: Recall that $f_{\mathbb{X}}(u_0) = \int_{-\infty}^{\infty} f_{\mathbb{X},\mathbb{Y}}(u_0, v)dv$ is just the area of the cross-section of the pdf solid along the vertical line at distance u_0 from the origin.



(a) (b)

Figure (a) above shows that there are only two cases when the line intersects the pdf solid and in both cases, the cross-section is a rectangle whose area can be determined without the formality of integration. The areas of the cross-sections are $\frac{1}{3} \times 1 = \frac{1}{3}$ and $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ also, and so we get

$$f_{\mathbb{X}}(u) = \begin{cases} \frac{1}{3}, & 1 \le u \le 2 \\ \frac{1}{3}, & 3 \le u \le 5 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{3}, & 1 \le u \le 2 \text{ or } 3 \le u \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

Those who insist on doing things "by the book" can proceed as follows.

For $1 \le u \le 2$, $f_{\mathbb{X}}(u) = \displaystyle\int_{-\infty}^{\infty} f_{\mathbb{X},\mathbb{Y}}(u,v)\,dv = \int_2^3 \frac{1}{3}\,dv = \frac{1}{3}v\Big|_2^3 = \frac{1}{3}(3-2) = \frac{1}{3}$.

For $3 \le u \le 5$, $f_{\mathbb{X}}(u) = \displaystyle\int_{-\infty}^{\infty} f_{\mathbb{X},\mathbb{Y}}(u,v)\,dv = \int_3^{\frac{7}{2}} \frac{2}{3}\,dv == \frac{2}{3}v\Big|_3^{\frac{7}{2}} = \frac{2}{3}\left(\frac{7}{2} - 3\right) = \frac{1}{3}$.

(c) [**6 points**] Find $P\{\mathbb{Y} > \mathbb{X}\}$.

Solution: The region of the plane where $\mathbb{Y} > \mathbb{X}$ is the half-plane above the diagonal line in Figure (b) above. Once again, we can determine the probability by computing the volume in that region. The solid with square base has area 1 and so contributes $\frac{1}{3} \times 1 = \frac{1}{3}$ while the little triangle has area $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ and therefore volume $\frac{1}{12}$ for a total volume of $\frac{5}{12}$, that is, $P\{\mathbb{Y} > \mathbb{X}\} = \frac{5}{12}$. Anti-segregationists (that is, those who believe in integration) can write

$$P\{\mathbb{Y} > \mathbb{X}\} = \int_1^2 \int_2^3 \frac{1}{3} dv du + \int_3^{\frac{7}{2}} \int_u^{\frac{7}{2}} \frac{2}{3} dv du = \frac{1}{3}(2-1)(3-2) + \frac{2}{3}\int_3^{\frac{7}{2}} \left(\frac{7}{2} - u\right) du$$

$$= \frac{1}{3} + \frac{2}{3} \left. \frac{\left(\frac{7}{2} - u\right)^2}{-2} \right|_3^{\frac{7}{2}} = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}.$$