**University of Illinois** Fall 2013

# ECE 313: Final Examination

Monday December 16, 2013 8:00 a.m. — 11:00 a.m.

1. **[64 points]**

Mark each statement as TRUE or FALSE by checking the appropriate box

Each correct answer will earn you 4 points if there is *some* indication of how you arrived at the answer checked. A *complete* solution is not expected, but a check mark in the correct box *without* **any** *indication of how you arrived at the answer* will be given no credit at all. Wrong answers (with or without justification) will also not be given any credit.

(a) Let $A$ and $B$ be events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $P(A \mid B)P(B) + P(A \mid B^c)P(B^c) = P(A)$. |
| ☐ | ☐ | $P(A \mid B)P(B) + P(A^c \mid B)P(B) = P(B)$. |
| ☐ | ☐ | $P(A \mid B^c)P(B^c) + P(A^c \mid B)P(B) = P(A \cup B) - P(A \cap B)$. |

(b) $\mathbb{X}$ is a *continuous* random variable with CDF $F_{\mathbb{X}}(u)$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | If $b > a$, then $F_{\mathbb{X}}(b) > F_{\mathbb{X}}(a)$. |
| ☐ | ☐ | If $F_{\mathbb{X}}(b) > F_{\mathbb{X}}(a)$, then $b > a$. |
| ☐ | ☐ | $F_{\mathbb{X}}(\alpha) = \frac{1}{2}$ for some $\alpha$, $-\infty < \alpha < \infty$. |

(c) $\mathbb{X}$ and $\mathbb{Y}$ are *independent* random variables.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | If $\mathbb{X} \sim \mathsf{Binomial}(m, p)$ and $\mathbb{Y} \sim \mathsf{Binomial}(n, p)$, then $\mathbb{X} + \mathbb{Y} \sim \mathsf{Binomial}(m + n, p)$. |
| ☐ | ☐ | If $\mathbb{X} \sim \mathsf{Geometric}(p)$ and $\mathbb{Y} \sim \mathsf{Geometric}(p)$, then $\mathbb{X} + \mathbb{Y} \sim \mathsf{Geometric}(p)$. |
| ☐ | ☐ | If $\mathbb{X} \sim \mathcal{N}(\mu_{\mathbb{X}}, \sigma_{\mathbb{X}}^2)$ and $\mathbb{Y} \sim \mathcal{N}(\mu_{\mathbb{Y}}, \sigma_{\mathbb{Y}}^2)$, then $\mathbb{X} - \mathbb{Y} \sim \mathcal{N}(\mu_{\mathbb{X}} - \mu_{\mathbb{Y}}, \sigma_{\mathbb{X}}^2 - \sigma_{\mathbb{Y}}^2)$. |

(d) $\mathbb{X}$ and $\mathbb{Y}$ are random variables with finite variances.

TRUE FALSE

☐ ☐ If $\mathsf{var}(\mathbb{X}+\mathbb{Y}) = \mathsf{var}(\mathbb{X}-\mathbb{Y})$, then $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$.

☐ ☐ If $\mathsf{var}(\mathbb{X}+\mathbb{Y}) = \mathsf{var}(\mathbb{X}-\mathbb{Y})$ then $\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated.*

☐ ☐ If $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$, then $\mathbb{X}+\mathbb{Y}$ and $\mathbb{X}-\mathbb{Y}$ are *uncorrelated.*

☐ ☐ If $\mathsf{var}(2\mathbb{X}+3\mathbb{Y}) = \mathsf{var}(3\mathbb{X}+2\mathbb{Y})$, then $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$.

(e) $\mathbb{X}$ is a continuous random variable with *standard deviation* 3 and *even* pdf, that is, $f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u), -\infty < u < \infty$.

TRUE FALSE

☐ ☐ $P\{\mathbb{X} \leq 6\} \geq \frac{7}{8} = 0.875$.

☐ ☐ If $\mathbb{X}$ is *Gaussian*, then $P\{\mathbb{X} \leq 6\} = 0.9772$.

☐ ☐ If $\mathbb{X}$ is *uniformly distributed*, then $P\{\mathbb{X} \leq 6\} < 1$.

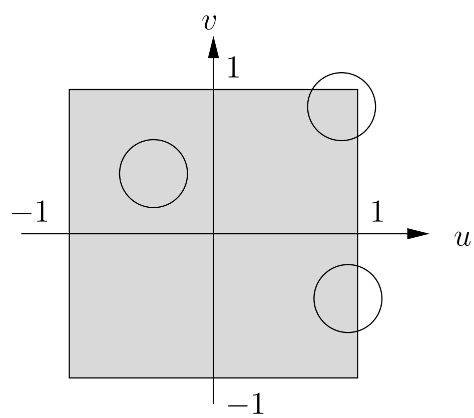
2. **[10 points]** We are given a coin that turns up Heads with unknown probability $p$. $n$ independent tosses of the coin result in $X$ Heads and we *estimate* $p$ as $\hat{p} = \frac{X}{n}$. Use the Chebychev Inequality to find the smallest $n$ such that the *confidence level* of the *confidence interval* $(\hat{p} - 0.1, \hat{p} + 0.1)$ is at least 99%.

3. **[21 points]** $\mathbb{Y}$ denotes a random variable with probability density function

$$f_{\mathbb{Y}}(v) = \begin{cases} 1 + v, & -1 \leq v \leq 0, \\ v, & 0 < v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $P\left\{|\mathbb{Y}| < \frac{1}{2}\right\}$, $P\left\{\mathbb{Y} > 0 \middle| \mathbb{Y} < \frac{1}{2}\right\}$, and $\mathsf{E}[\mathbb{Y}]$.

4. **[21 points]** The random point $(\mathbb{X}, \mathbb{Y})$ is uniformly distributed on the interior of a square of side 2 centered at the origin. Consider a circle of radius $r < 1$ centered at $(\mathbb{X}, \mathbb{Y})$, and let $\mathbb{Z} \in \{0, 1, 2\}$ denote the number of sides of the square that are crossed by the circle, as illustrated in the figure below.




Find the pmf of $\mathbb{Z}$ in terms of $r$.

5. **[10 points]** Suppose $(X, Y)$ is uniformly distributed over $[0, 1]^2$. Let $W = \dfrac{X}{X + Y}$.
Find $F_W(\alpha)$, the CDF of $W$. To obtain full credit, you mist specify the value of $F_W(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.
Hint: consider the cases $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$ separately.

6. **[25 points]** A fair coin and a biased coin with $P(\text{Heads}) = \frac{3}{4}$ are in a pocket. One coin is drawn at random from the pocket and flipped independently three times.

   (a) **[10 points]** What is the probability that heads show up on all three flips?

   (b) **[15 points]** Given that a total of two heads show up on the three flips, what is the probability that the coin is fair?

7. **[24 points]** Let $X$ and $Y$ be discrete random variables with joint pmf

$$p_{X,Y}(i, j) = \begin{cases} p^2(1-p)^{j-2} & i \in \{1, 2, 3, \ldots\} \textit{ and } j \in \{i+1, i+2, i+3, \ldots\} \\ 0 & \text{else,} \end{cases}$$

(a) **[7 points]** $X$ and $Y$ independent. State True or False. Justify your answer.

TRUE FALSE

☐ ☐

(b) **[10 points]** Obtain the marginal pmf $p_X(i)$ for all values of $i$. Simplify as much as you can.

(c) **[7 points]** ) Find $P\{2Y^2 - X \geq 0\}$. A numeric answer is expected.

8. **[10 points]** Three draws are made *with replacement* from a bag containing 7 tokens numbered 1 through 7. What is the probability that the sum of the three numbers drawn is 9?

9. **[10 points]** You hold the Q in a Scrabble game, and need a U to be able to make a word. There are 9 letters left in the bag, 2 of which are U's. If you draw 4 letters at random from the bag, what is the probability that you will get at least one U?

10. **[30 points]** Suppose $X, Y,$ and $Z$ are mutually independent random variables that all have the same mean 4 and same variance 9. Find the numerical values of the quantities indicated.

(a) **[5 points]** $\mathsf{var}(X + 2Y + 3Z + 5) =$

(b) **[5 points]** $\mathsf{cov}(X + Y, X + Z) =$

(c) **[7 points]** $\mathsf{E}[X^2Y^2] =$

(d) **[13 points]** Let $U = X + Y$ and $V = Y + Z$. Find the linear minimum mean square estimator of U given V = 10.