Geometric and Modified Geometric Distributions – Class Project 2

ECE 313

Probability with Engineering Applications

Lecture 9

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Today's topics

- Geometric and Modified Geometric Distributions
- Class Project 2

- Announcements:
  - Homework 4 will be posted this Thursday, 09/26
  - Mini Project 2 will be handed out next Tuesday, 10/1

  - Midterm, October 22, in class
    11:00am – 12:30pm
    Will start promptly at 11:00am. Please be on time

Discrete Distributions the Geometric pmf

- Consider a sequence of Bernoulli trials, we count the **number of trials until the first "success"** occurs. (instead of the usual: *number of successes in n trials*)
- Let 0 denote a failure and let 1 denote a success, then the sample space consists of the set of all binary strings with an arbitrary number of 0's followed by a single 1.
- $S = \{0^{i-1} 1 \mid i = 1, 2, 3, \ldots\}$ (S is a countably infinite set)
- Define a random variable Z on this sample space so that the value assigned to the sample point $0^{i-1} 1$ is i.
- Thus Z is a random variable with image {1, 2,...}, which is a countably infinite set

Discrete Distributions the Geometric pmf (cont.)

- To find the pmf of Z note that the event [Z = i] occurs if and only if we have a sequence of (i – 1) failures followed by one success - a sequence of independent Bernoulli trials with the probability of success equal to p and failure q.
- Hence, we have

$$p_Z(i) = q^{i-1}p = p(1-p)^{i-1} \quad \text{for } i = 1, 2, \ldots, \qquad \text{(A)}$$

- where q = 1 - p.
- Using the formula for the sum of a geometric series, we have:

$$\sum_{i=1}^{\infty} p_Z(i) = \sum_{i=1}^{\infty} pq^{i-1} = \frac{p}{1-q} = \frac{p}{p} = 1$$

Discrete Distributions the Geometric pmf (cont.)

- Any random variable with the image {1, 2,...} and pmf given by the formula of the form of equation (A) is said to have a *geometric distribution* and the function given by (A) is called a *geometric pmf* with parameter p.

- The corresponding CDF is:

$$F_Z(t) = \sum_{i=1}^{\lfloor t \rfloor} p(1-p)^{i-1} = 1-(1-p)^{\lfloor t \rfloor} \quad \text{for } t \geq 0$$

Discrete Distributions the Modified Geometric pmf (cont.)

- The random variable Z – a geometric r.v. - counts the total number of trials up to and including the first success.
- We are often interested in counting the number f failures before the first success.
- Let this number be a random variable X with the image {0, 1, 2,...}. Clearly, $Z = X + 1$.

Discrete Distributions the Modified Geometric pmf (cont.)

- The random variable X is said to have a modified geometric pmf, specify by

$$p_X(i) = p(1-p)^i \qquad \text{for } i = 0, 1, 2, \ldots,$$

- The corresponding distribution function is:

$$F_X(t) = \sum_{i=0}^{\lfloor t \rfloor} p(1-p)^i = 1-(1-p)^{\lfloor t+1 \rfloor} \qquad \text{for } t \geq 0$$

Geometric Distribution Examples

- Examples where the geometric distribution occurs include:

1. A series of components made by a certain manufacturer. The probability the ith item is defective one is given by the geometric pmf.

2. Consider the operation of a time-sharing computer system with a fixed time-slice. The pmf of the random variable denoting the number of time slices needed to complete the execution of a program is given by geometric pmf.

Geometric Distribution Examples

3. Consider the program segment consisting of a ***while*** loop:

- **while ¬ B do S**

- If the successive test of the Boolean expression B are independent , then the number of times the body (or the statement-group S) of the loop is executed will be a random variable having a modified geometric distribution with parameter p (probability the B is true) – no. of failures until the first success.

4. Consider a **repeat** loop

- **repeat S until B**

- The number of tries until B (success) is reached will be a geometrically distributed random variable with parameter p.

Class Project 2 – Part 1

- Consider the experiment of tossing two dice. The sample space is S = {(i,j)|1 ≤ i,j ≤ 6}. Assume all the sample points have the equal probability of 1/36. Let:

A = "The first die results in a 1, 2, or 6."

B = "The first die results in a 3, 4, or 5."

C = "The sum of the two faces is 9."

- Write the events *A, B,* and *C* and calculate their probabilities
- Write the events $A \cap B; A \cap C; B \cap C; A \cap B \cap C;$ and their probabilities
- Determine if: $P(A \cap B) = P(A)P(B)$

$$P(A \cap C) = P(A)P(C)$$

- Determine $P(A \cap B \cap C) = P(A)P(B)P(C)$
- What you can say about exclusiveness and independence of these events?

Class Project 2 – Part 1

- Repeat for:

  A = "The first die results in a 1 or 2"

  B = "The second die results in a 4, 5"

  C = "The sum of the two faces is 6."

- What you can say about exclusiveness and independence of these events?

- Refer to the course website (lectures) for the solution for Class Project 2.

Some Important Points about the Concept of Independence

- If A and B are two mutually exclusive events, then $A \bigcap B = \emptyset$, which implies $P(A \bigcap B) = 0$.
- Now, if they are independent as well, we have: $P(A \bigcap B) = P(A)P(B)$, then either P(A)=0 or P(B)=0.

- If the events A and B are independent, and the events B and C are independent, then events A and C need not be independent (i.e., independence is not a transitive relation).
- If the events A1, A2, …, An are such that every pair is independent, then they are called **pairwise independent**. It does not follow that the list of events is **mutually independent**.

Some Important Points about the Concept of Independence (cont.)

- If the events A and B are independent, then so are events $\overline{A}$ and B, events A and $\overline{B}$, and events $\overline{A}$ and $\overline{B}$. Note that $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive events whose union is B, i.e.,

  $$P(B) = P(A \cap B) + P(\overline{A} \cap B) = P(A)P(B) + P(\overline{A} \cap B),$$

  since A and B are independent.

  This implies

  $$P(\overline{A} \cap B) = P(B) - P(A)P(B) = P(B)[1 - P(A)] = P(B)(P(\overline{A}).$$

- The independence of A and $\overline{B}$ and $\overline{A}$ and $\overline{B}$ can be shown similarly.
- The concept of independence of two events can be extended to a list of *n* events.

Class Project 2 – Part 2

- A telephone call may pass through a series of trunks before reaching its destination. If the destination is within the caller's own local exchange, then no trunks will be used.
- Assume *p* is the probability of reaching to destination.
- Let *X* the number of trunks used to reach to a destination, which is a modified geometric random variable with parameter *p*.
- Define *Z* to be the number of trunks used for a call directed to a destination outside the caller's local exchange.
- Given that a call requires at least three trunks, what is the conditional pmf of the number of trunks required ?

- Find the set of values that X might take and write its pmf.
- Find the set of values that Z might take and write its pmf.
- Write the expression for conditional probability and simplify it as much as you can.

Class Project 2 - Solution

- X can take any values of {0, 1, 2, 3, …}
- The pmf of *X* is: $P_X(i) = P(X = i) = p(1-p)^i, \; i = 0,1,2,....$
- Z is the number of trunks used for a call directed to a destination outside the caller's local exchange, so at least one trunk will be used and Z takes values of {1, 2, 3, …}
- Since X has modified geometric distribution, then Z has geometric distribution with pmf: $P_Z(k) = P(Z = k) = p(1-p)^{k-1}, k = 1,2,...$
- The conditional probability is required for

$$P(Z = k \mid Z \geq 3) = \frac{P(Z = k \; and \; Z \geq 3)}{P(Z \geq 3)}$$

$$= \begin{cases} \dfrac{p(1-p)^{k-1}}{(1-p)^{3-1}} & , \quad k \geq 3 \\ 0 & , otherwise \end{cases}$$

$$or = p(1-p)^{k-3}, k \geq 3.$$

- Note that we are not shifting the origin of measurement and therefore the use of the memoryless property to obtain the answer is incorrect.